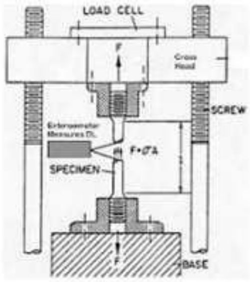
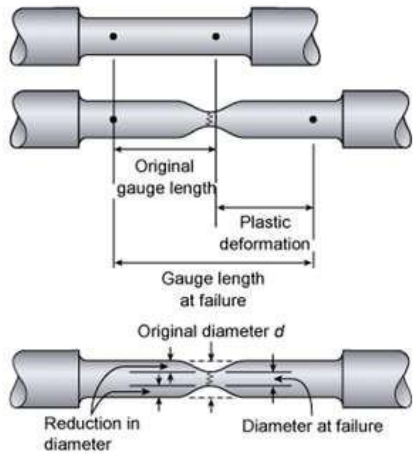
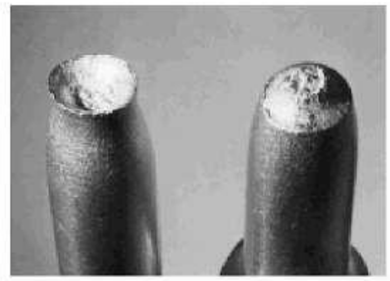
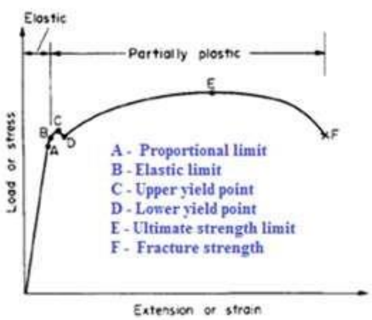
Module 1 : Simple Stresses and Strains







Cup and Cone fracture



Introduction, Definition and concept and of stress and strain. Hooke's law, Stress-Strain diagrams for ferrous and non-ferrous materials, factor of safety, Elongation of tapering bars of circular and rectangular cross sections, Elongation due to self-weight. Saint Venant's principle, Compound bars, Temperature stresses, Compound section subjected to temperature stresses, state of simple shear, Elastic constants and their relationship.

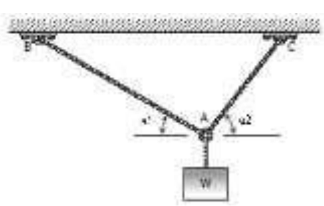
Introduction

In civil engineering structures, we frequently encounter structural elements such as *tie members, cables, beams, columns and struts* subjected to external actions called *forces or loads.* These elements have to be designed such that they have adequate *strength, stiffness and stability*.

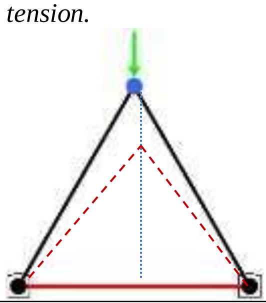
The *strength* of a structural component is its ability to withstand applied forces without failure and this depends upon the *sectional dimensions and material characteristics*. For instance a steel rod can resist an applied tensile force more than an aluminium rod with similar diameter. Larger the sectional dimensions or stronger is the material greater will be the force carrying capacity.

Stiffness influences the deformation as a consequence of *stretching, shortening, bending, sliding, buckling, twisting and warping* due to applied forces as shown in the following figure. In a *deformable* body, the distance between two points changes due to the action of some kind of forces acting on it.

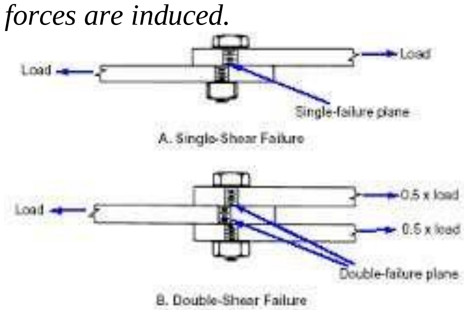
A weight suspended by two cables causes stretching of the cables. Cables are in *axial tension.*



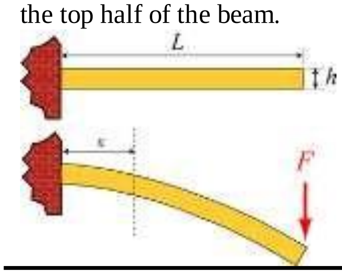
Inclined members undergo shortening, and stretching will be induced in the horizontal member. Inclined members are in *axial compression* and horizontal member is in *axial tension.*



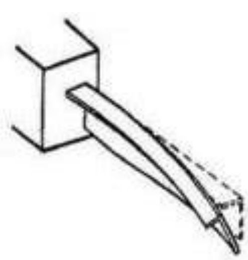
Bolt connecting the plates is subjected to sliding along the failure plane. *Shearing forces are induced.*



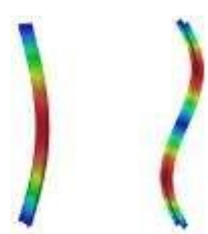
Cantilever beam subjected to *bending* due to transverse loads results in shortening in the bottom half and stretching in the top half of the beam.



Cantilever beam subjected to twisting and warping due to *torsional moments.*



Buckling of long compression members due to *axial load.*



Such deformations also depend upon *sectional dimensions, length and material characteristics*. For instance a steel rod undergoes less of stretching than an aluminium rod with similar diameter and subjected to same tensile force.

Stability refers to the ability to maintain its original configuration. This again depends upon *sectional dimensions, length and material characteristics.* A steel rod with a larger length will buckle under a compressive action, while the one with smaller length can remain stable even though the sectional dimensions and material characteristics of both the rods are same.

The subject generally called *Strength of Materials* includes the study of the distribution of internal forces, the stability and deformation of various elements. It is founded both on the results of experiments and the application of the principles of mechanics and mathematics. The results obtained in the subject of strength of materials form an important part of the basis of scientific and engineering designs of different structural elements. Hence this is treated as subject of fundamental importance in design engineering. The study of this subject in the context of civil engineering refers to various methods of analyzing deformation behaviour of structural elements such as plates, rods, beams, columns, shafts etc.,.

Concepts and definitions

A load applied to a structural member will induce internal forces within the member called *stress resultants* and if computed based on unit cross sectional area then they are termed as *intensity of stress or simply stress* in the member.

The stresses induced in the structural member will cause different types of deformation in the member. If such deformations are computed based on unit dimensions then they are termed as *strain* in the member.

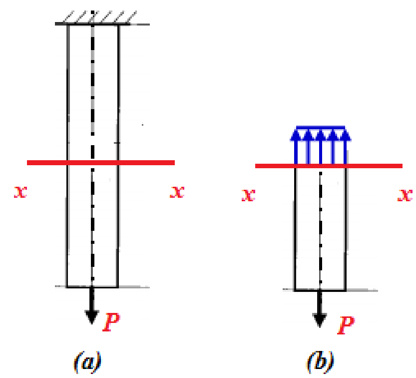
The stresses and strains that develop within a structural member must be calculated in order to assess its strength, deformations and stability. This requires a complete description of the geometry, constraints, applied loads and the material properties of the member.

The calculated stresses may then be compared to some measure of the strength of the material established through experiments. The calculated deformations in the member may be compared with respect limiting criteria established based on experience. The calculated buckling load of the member may be compared with the applied load and the safety of the member can be assessed.

It is generally accepted that analytical methods coupled with experimental observations can provide solutions to problems in engineering with a fair degree of accuracy. Design solutions are worked out by a proper analysis of deformation of bodies subjected to surface and body forces along with material properties established through experimental investigations.

Simple Stress

Consider the suspended bar of original length $\mathbf{L_0}$ and uniform cross sectional area $\mathbf{A_0}$ with a force or load **P** applied to its end as shown in the following figure (a). Let us imagine that the bar is cut in to two parts by a section ***x-x*** and study the equilibrium of the lower portion of the bar as shown in figure (b). At the lower end, we have the applied force P



It can be noted that, the external force applied to a body in equilibrium is reacted by internal forces set up within the material. If a bar is subjected to an axial tension or compression, **P,** then the internal forces set up are distributed uniformly and the bar is said to be subjected to a *uniform direct or normal or simple stress*. The stress being defined as

$$stress\ (\sigma) = \frac{Load\ (P)}{Sectional\ Area\ (A)}$$

Note

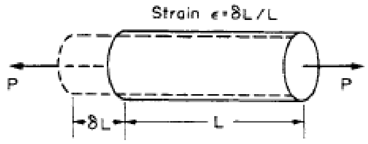
i. This is expressed as N/mm^2 or MPa.

ii. Stress may thus be compressive or tensile depending on the nature of the load.

iii. In some cases the stress may vary across any given section, and in such cases the stress at any point is given by the limiting value of $\delta P/\delta A$ as δA tends to zero.

Simple Strain

If a bar is subjected to a direct load, and hence a stress, the bar will change in length. If the bar has an original length L and changes in length by an amount δL as shown below,



then the strain produced is defined as follows:

$$strain\ \varepsilon = \frac{change\ in\ length\ (\delta L)}{original\ length\ (L)}$$

This strain is also termed as *longitudinal strain* as it is measured in the direction of application of load.

Note:

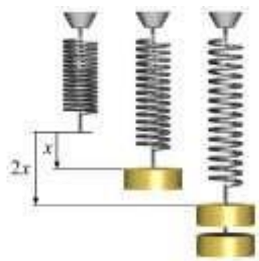
iv. Strain is thus a measure of the deformation of the member. It is simply a ratio of two quantities with the same units. It is non-dimensional, i.e. it has no units.

v. The deformations under load are very small. Hence the strains are also expressed as *strain x 10* $^{-6}$. In such cases they are termed as *microstrain* ($\mu\varepsilon$).

vi. Strain is also expressed as a percentage strain : ε (%) = (δL/L)100.

Elastic limit – *Hooke's law*

A structural member is said to be within elastic limit, if it returns to its original dimensions when load is removed. Within this load range, the deformations are proportional to the loads producing them. Hooke's law states that, "*the force needed to extend or compress a spring by some distance is proportional to that distance*". This is indicated in the following figure.



Since loads are proportional to the stresses they produce and deformations are proportional to the strains, the Hooke"s law also implies that, "*stress is proportional to strain within elastic limit*".

$$stress\ (\sigma) \propto strain(\varepsilon) \quad \text{or} \quad \sigma/\varepsilon = constant$$

This law is valid within certain limits for most ferrous metals and alloys. It can even be assumed to apply to other engineering materials such as concrete, timber and non-ferrous alloys with reasonable accuracy.

The law is named after 17th-century British physicist Robert Hooke. He first stated the law in 1676 as a Latin anagram. He published the solution of his anagram in 1678 as: "uttensio, sic vis" ("as the extension, so the force" or "the extension is proportional to the force").

Modulus of elasticity or Young's modulus

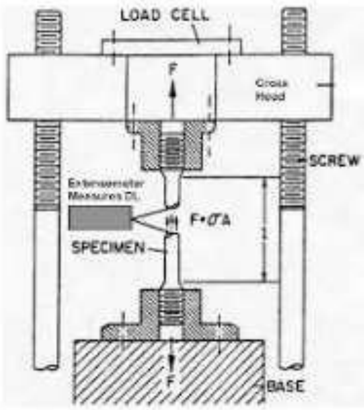
Within the elastic limits of materials, i.e. within the limits in which Hooke's law applies, it has been found that stress/strain = constant. This is termed the *modulus of elasticity or Young's modulus*. This is usually denoted by letter E and has the same units of stress. With $\sigma = P/A$ and $\varepsilon = \delta L/L$, the following expression for E can be derived.

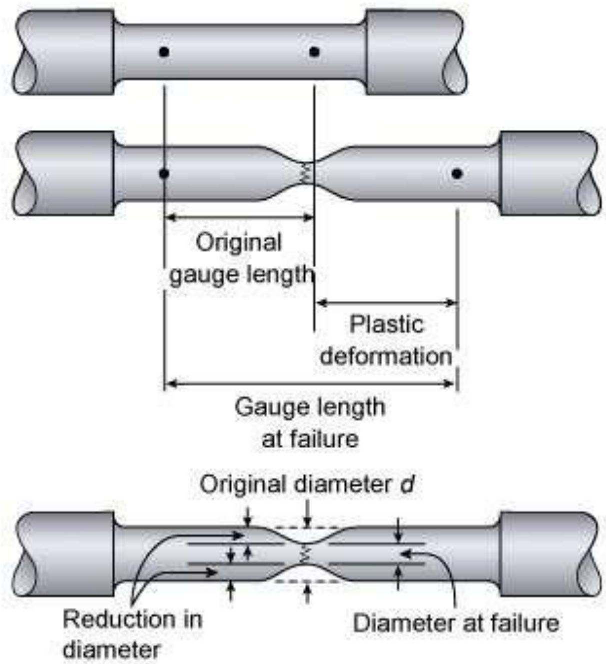
$$E = \frac{\sigma}{\varepsilon} = \frac{P}{A}\frac{L}{\delta L}$$

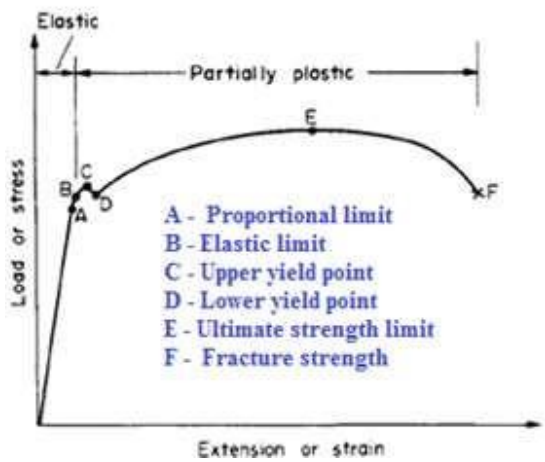
Young's modulus E is generally assumed to be the same in tension or compression and for most engineering materials has a high numerical value. Typically, E = 200000 MPa for steel. This is determined by conducting tension or compression test on specimens in the laboratory.

Tension test

In order to compare the strengths of various materials it is necessary to carry out some standard form of test to establish their relative properties. One such test is the standard tensile test. In this test a circular bar of uniform cross-section is subjected to a gradually increasing tensile load until failure occurs. Measurements of the change in length of a selected gauge length of the bar are recorded throughout the loading operation by means of extensometers. A graph of load against extension or stress against strain is produced.

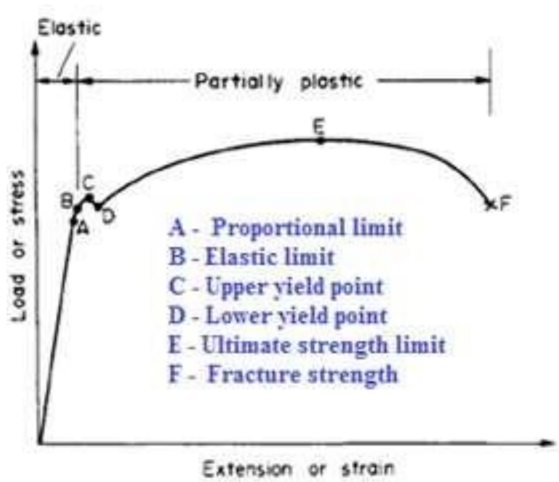






Stress – Strain diagrams for ferrous metals

The typical graph for a test on a mild (low carbon) steel bar is shown in the figure below. Other materials will exhibit different graphs but of a similar general form. Following salient points are to be noted:



i. In the initial stages of loading it can be observed that Hooke's law is obeyed, i.e. the material behaves elastically and stress is proportional to strain. This is indicated by the straight-line portion in the graph up to point A. Beyond this, some nonlinear nature of the graph can be seen. Hence this point (A) is termed the *limit of proportionality*. This region is also called *linear elastic range* of the material.

ii. For a small increment in loading beyond A, the material may still be elastic. Deformations are completely recovered when load is removed but Hooke's law does not apply. The limiting point B for this condition is termed the *elastic limit*. This region refers to *nonlinear elastic range*. It is often assumed that points A and B are coincident.

iii. Beyond the elastic limit (A or B), *plastic deformation* occurs and strains are not totally recoverable. Some *permanent deformation* or *permanent set* will be there when the specimen is unloaded. Points C, is termed as the *upper yield point*, and D, as the *lower yield point*. It is often assumed that points C and D are coincident. Strength corresponding to *this* point is termed as the *yield strength* of the material. Typically this strength corresponds to the load carrying capacity.

iv. Beyond point (C or D), strain increases rapidly without proportionate increases in load or stress. The graph covers a much greater portion along the strain axis than in the elastic range of the material. The capacity of a material to allow these large plastic deformations is a measure of *ductility* of the material.

v. Some increase in load is required to take the strain to point E on the graph. Between D and E the material is said to be in the *elastic-plastic state*. *S*ome of the section remaining elastic and hence contributing to recovery of the original dimensions if load is removed, the remainder being plastic.

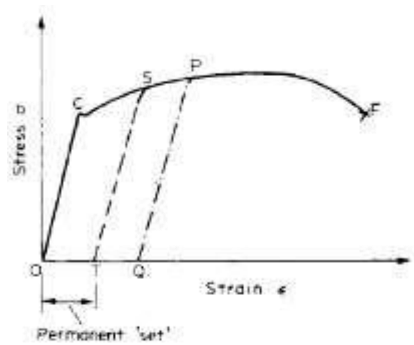
vi. Beyond E, the cross-sectional area of the bar begins to reduce rapidly over a relatively small length. This result in the formation of *necking* accompanied with reduction in load and *fracture (cup and cone)* of the bar eventually occurs at point F.

vii. The nominal stress at failure, termed the *maximum or ultimate tensile stress*, is given by the load at E divided by the original cross-sectional area of the bar. This is also known as the *ultimate tensile strength* of the material.

viii. Owing to the large reduction in area produced by the necking process the actual stress at fracture is often greater than the ultimate tensile strength. Since, however, designers are interested in maximum loads which can be carried by the complete cross-section, the stress at fracture is not of any practical importance.

Influence of Repeated loading and unloading on yield strength

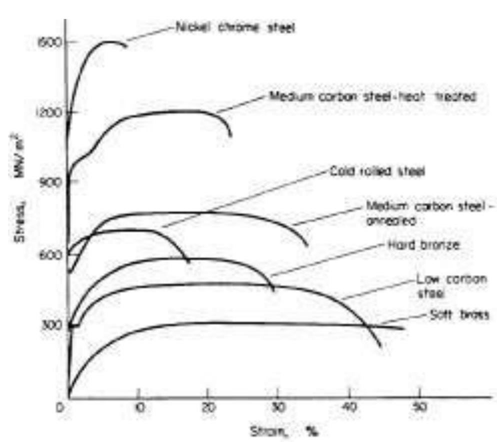
If load is removed from the test specimen after the yield point C has been passed, e.g. to some position S, as shown in the adjoining figure the unloading line ST can, for most practical purposes, be taken to be linear. A second load cycle, commencing with the permanent elongation associated with the strain OT, would then follow the line TS and continue along the previous



curve to failure at F. It can be observed, that the repeated load cycle has the effect of increasing the elastic range of the material, i.e. raising the effective yield point from C to S. However, it is important to note that the tensile strength is unaltered. The procedure could be repeated along the line PQ, etc., and the material is said to have been work hardened. Repeated loading and unloading will produce a yield point approaching the ultimate stress value but the elongation or strain to failure will be very much reduced.

Non Ferrous metals

Typical stress-strain curves resulting from tensile tests on other engineering materials are shown in the following figure.



For certain materials, for example, high carbon steels and non-ferrous metals, it is not possible to detect any difference between the upper and lower yield points and in some cases yield point may not exist at all. In such cases a proof stress is used to indicate the onset of plastic strain. The 0.1% proof stress, for example, is that stress which, when removed, produces a permanent strain of 0.1% of the original gauge length as shown in the following figure.

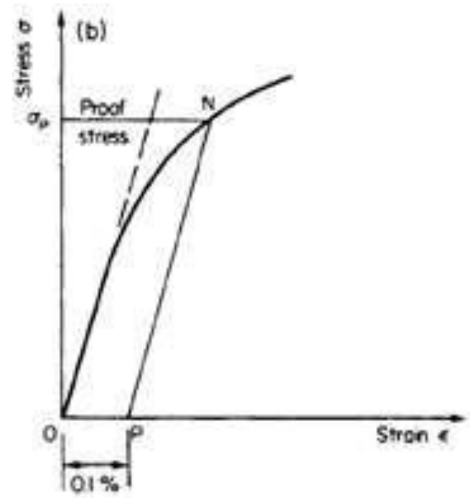
The 0.1% proof stress can be determined from the tensile test curve as listed below.

vii. Mark the point P on the strain axis which is equivalent to 0.1% strain.

viii. From P draw a line parallel with the initial straight line portion of the tensile test curve to cut the curve in N.

ix. The stress corresponding to N is then the 0.1% proof stress.

x. A material is considered to satisfy its specification if the permanent set is no more than 0.1% after the proof stress has been applied for 15 seconds and removed.



Allowable working stress-factor of safety

The most suitable strength criterion for any structural element under service conditions is that some maximum stress must not be exceeded such that plastic deformations do not occur. This value is generally known as the *maximum allowable working stress*. Because of uncertainties of loading conditions, design procedures, production methods etc., it is a common practice to introduce a *factor of safety* into structural designs. This is defined as follows:

$$\text{Factor of safety} = \frac{\text{Yield stress (or proof stress)}}{\text{Allowable woking stress}}$$

Ductile materials

The capacity of a material to allow large extensions, i.e. the ability to be drawn out plastically, is termed its *ductility*. A quantitative value of the ductility is obtained by measurements of the percentage elongation or percentage reduction in area as defined below.

$$\%\ elongation = \frac{increase\ in\ gauge\ length\ to\ fracture}{original\ gauge\ length} \times 100$$

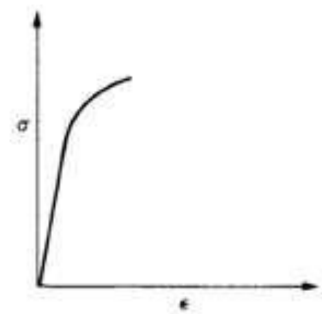
$$\%\ reduction\ in\ area = \frac{cross\ sectional\ area\ of\ necked\ portion}{original\ area} \times 100$$

Note:

A property closely related to ductility is malleability, which defines a material's ability to be hammered out into thin sheets. Malleability thus represents the ability of a material to allow permanent extensions in all lateral directions under compressive loadings.

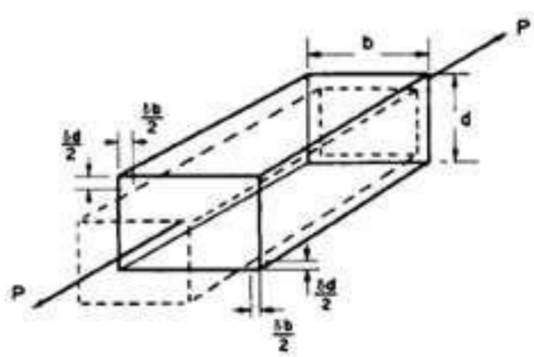
Brittle materials

A brittle material is one which exhibits relatively small extensions to fracture so that the partially plastic region of the tensile test graph is much reduced. There is little or no necking at fracture for brittle materials. Typical tensile test curve for a brittle material could well look like the one shown in the adjoining figure.



Lateral strain and Poisson's ratio

Till now we have focused on the longitudinal strain induced in the direction of application of the load. It has been observed that deformations also take place in the lateral direction. Consider the rectangular bar shown in the figure below and subjected to a tensile load.



Under the action of this load the bar will increase in length by an amount δL giving a longitudinal strain in the bar: **$\varepsilon_L = \delta L/L$.** The bar will also exhibit, however, a reduction in dimensions laterally, i.e. its breadth and depth will both reduce. The associated lateral strains will both be equal, and are of opposite sense to the longitudinal strain. These are computed as :

$\varepsilon_{lat} = \delta b/b = \delta d/d.$

It has been observed that within the elastic range the ratio of the lateral and longitudinal strains will always be constant. This ratio is termed *Poisson's ratio (ν)*.

$$\nu = \frac{\varepsilon_{lat}}{\varepsilon_L}$$

The above equation can also be written as :

$$\varepsilon_{lat} = \nu\varepsilon_L = \nu\frac{\sigma}{E}$$

For most of the engineering materials the value of ν is found to be between 0.25 and 0.33.

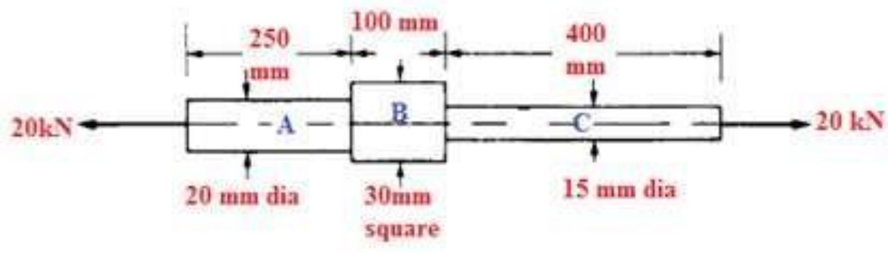
Example 1

A bar of a rectangular section of 20 mm × 30 mm and a length of 500 mm is subjected to an axial compressive load of 60 kN. If E = 102 kN/mm^2 and υ = 0.34, determine the changes in the length and the sides of the bar.

- Since the bar is subjected to compression, there will be decrease in length, increase in breadth and depth. These are computed as shown below
- L = 500 mm, b = 20 mm, d = 30 mm, P = 60 x1000 = 60000 N, E = 102000 N/mm^2
- Cross-sectional area A = 20 × 30 = 600 mm^2
- Compressive stress σ = P/A = 60000/600 = 100 N/mm^2
- Longitudinal strain ε_L = σ/E = 100/102000 = 0.00098
- Lateral strain ε_{lat} = ν ε_L = 0.34 x 0.00098 = 0.00033
- Decrease in length δL = ε_L L = 0.00098 x 500 = 0.49 mm
- Increase in breadth δb = ε_{lat} b = 0.00033 x 20 = 0.0066 mm
- Increase in depth δd = ε_{lat} d = 0.00033 x 30 = 0.0099 mm

Example 2

Determine the stress in each section of the bar shown in the following figure when subjected to an axial tensile load of 20 kN. The central section is of square cross-section; the other portions are of circular section. What will be the total extension of the bar? For the bar material E = 210000MPa.



The bar consists of three sections with change in diameter. Loads are applied only at the ends. The stress and deformation in each section of the bar are computed separately. The total extension of the bar is then obtained as the sum of extensions of all the three sections. These are illustrated in the following steps.

The bar is in equilibrium under the action of applied forces

Stress in each section of bar = P/A and P = 20000N

xi. Area of Bar A = π x $20^2/4$ = 314.16 mm^2

ii. Stress in Bar A : σ_A = 20000/ 314.16 = 63.66MPa

iii. Area of Bar B = 30 x30 = 900 mm^2

iv. Stress in Bar B : σ_B = 20000/ 900 = 22.22MPa

v. Area of Bar C = π x $15^2/4$ = 176.715 mm^2

vi. Stress in Bar C : σ_C = 20000/ 176.715 = 113.18MPa

Extension of each section of bar = σL/E and E = 210000 MPa

i. Extension of Bar A = 63.66 x 250 / 210000= 0.0758 mm

ii. Extension of Bar B = 22.22 x 100 / 210000= 0.0106 mm

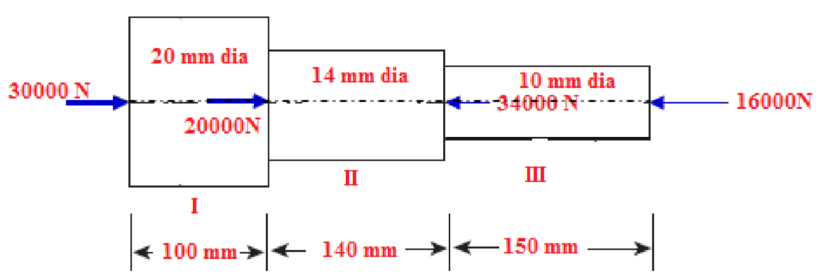
iii. Extension of Bar C = 113.18 x 400 / 210000= 0.2155 mm

Total extension of the bar = **0. 302mm**

Example 3

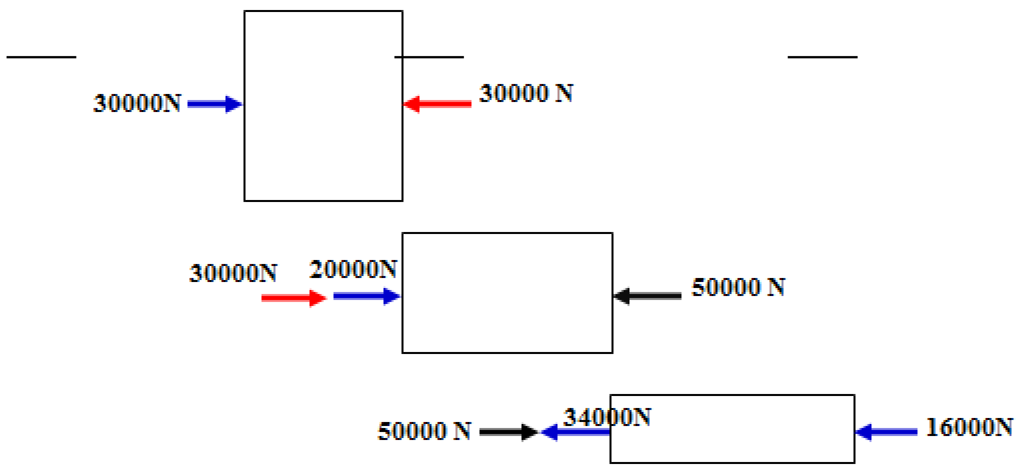
Determine the overall change in length of the bar shown in the figure below with following data:

E = 100000 N/mm^2



The bar is with varying cross-sections and subjected to forces at ends as well as at other interior locations. It is necessary to study the equilibrium of each portion separately and compute the change in length in each portion. The total change in length of the bar is then obtained as the sum of extensions of all the three sections as shown below.

Forces acting on each portion of the bar for equilibrium



Sectional Areas

$$A_I = \frac{\pi \times 20^2}{4} = 314.16\, mm^2\ ;\ A_{II} = \frac{\pi \times 14^2}{4} = 153.94\, mm^2;\ A_{III} = \frac{\pi \times 10^2}{4} = 78.54\, mm^2$$

Change in length in Portion I

Portion I of the bar is subjected to an axial compression of 30000N. This results in *decrease* in length which can be computed as

$$\delta L_I = \frac{P_I L_I}{A_I E} = \frac{30000 \times 100}{314.16 \times 100000} = 0.096\, mm$$

Change in length in Portion II

Portion II of the bar is subjected to an axial compression of 50000N (30000 + 20000). This results in *decrease* in length which can be computed as

$$\delta L_I = \frac{P_{II} L_{II}}{A_{II} E} = \frac{50000 \times 140}{153.94 \times 100000} = 0.455 mm$$

Change in length in Portion III

Portion III of the bar is subjected to an axial compression of (50000 – 34000) = 16000N. This results in *decrease* in length which can be computed as

$$\delta L_I = \frac{P_{III} L_{III}}{A_{III} E} = \frac{16000 \times 150}{78.54 \times 100000} = 0.306mm$$

Since each portion of the bar results in decrease in length, they can be added without any algebraic signs.

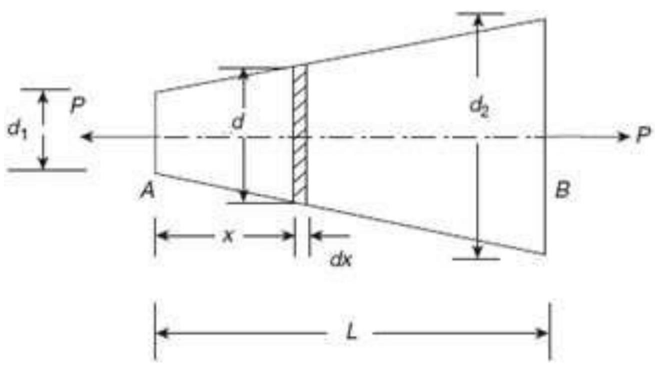
Hence Total decrease in length = 0.096 + 0.455 + 0.306 = **0.857mm**

Note:

For equilibrium, if some portion of the bar may be subjected to tension and some other portion to compression resulting in increase or decrease in length in different portions of the bar. In such cases, the total change in length is computed as the sum of change in length of each portion of the bar with proper algebraic signs. Generally positive sign (+) is used for increase in length and negative sign (-) for decrease in length.

Elongation of tapering bars of circular cross section

Consider a circular bar uniformly tapered from diameter **d_1** at one end and gradually increasing to diameter **d_2** at the other end over an axial length **L** as shown in the figure below.



Since the diameter of the bar is continuously changing, the elongation is first computed over an elementary length and then integrated over the entire length. Consider an elementary strip of diameter ***d*** and length ***dx*** at a distance of ***x*** from end ***A***.

Using the principle of similar triangles the following equation for d can be obtained

$$d = d_1 + \frac{d_2 - d_1}{L}x = d_1 + kx \text{ , where } k = \frac{d_2 - d_1}{L}$$

Cross–sectional area of the bar at x : $A_x = \frac{\pi (d_1 + kx)^2}{4}$

Axial stress at x: $\sigma_x = \frac{P}{A_x} = \frac{4P}{\pi (d_1 + kx)^2}$

Change in length over dx : $\delta dx = \frac{\sigma_x \, dx}{E} = \frac{4P \, dx}{\pi E \, (d_1 + kx)^2}$

Total change in length: $\delta L = \int_0^L \frac{4P\,dx}{\pi E\,(d_1+kx)^2} = \frac{4P}{\pi E}\left[\frac{(d_1+kx)^{-1}}{-k}\right]_0^L$

After rearranging the terms: $\delta L = -\frac{4P}{\pi Ek}\left[\frac{1}{(d_1+kx)}\right]_0^L$

Upon substituting the limits : $\delta L = -\frac{4P}{\pi Ek}\left[\frac{1}{(d_1+kL)} - \frac{1}{d_1}\right]$

After rearranging the terms: $\delta L = \frac{4P}{\pi Ek}\left[\frac{1}{d_1} - \frac{1}{(d_1+kL)}\right]$

But $(d_1 + kL) = d_1 + \frac{d_2 - d_1}{L} L = d_2$

With the above substitution: $\delta L = \frac{4P}{\pi Ek}\left[\frac{1}{d_1} - \frac{1}{d_2}\right] = \frac{4P}{\pi Ek}\left[\frac{d_2 - d_1}{d_1 d_2}\right]$

Substituting for $k = \frac{d_2 - d_1}{L}$ in the above expression, following equation for elongation of tapering bar of circular section can be obtained

$$\text{Total change in length: } \delta L = \frac{4P\,L}{\pi E\,d_1 d_2}$$

Example 4

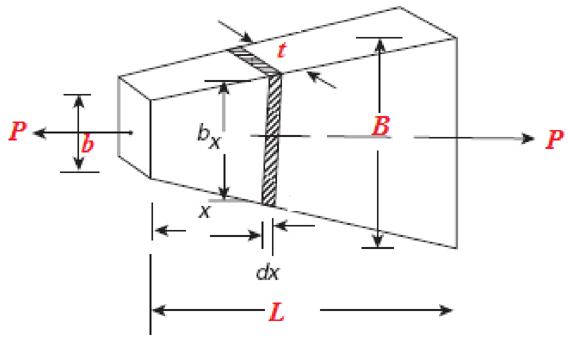
A bar uniformly tapers from diameter 20 mm at one end to diameter 10 mm at the other end over an axial length 300 mm. This is subjected to an axial compressive load of 7.5 kN. If *E* = 100 kN/mm^2, determine the maximum and minimum axial stresses in bar and the total change in length of the bar.

P = 7500 N, E = 100000 N/mm^2, d_1 = 10mm, d_2 = 20mm, L = 300mm

- Minimum compressive stress occurs at d_2 = 20mm as the sectional area is maximum.
- Area at $d_2 = \frac{\pi \times 20^2}{4} = 314.16mm^2$
- $\sigma_{min} = \frac{7500}{314.16} = 23.87MPa$
- Maximum compressive stress occurs at d_1 = 10mm as the sectional area is minimum.
- Area at $d_1 = \frac{\pi \times 10^2}{4} = 78.54mm^2$
- $\sigma_{min} = \frac{7500}{78.54} = 95.5MPa$
- Total decrease in length: $\delta L = \frac{4P\,L}{\pi E\,d_1 d_2} = \frac{4 \times 7500 \times 300}{\pi \times 100000 \times 10 \times 20} = 0.143$mm

Elongation of tapering bars of rectangular cross section

Consider a bar of same thickness **t** throughout its length, tapering uniformly from a breadth **B** at one end to a breadth **b** at the other end over an axial length **L**. The flat is subjected to an axial force **P** as shown in the figure below.



Since the breadth of the bar is continuously changing, the elongation is first computed over an elementary length and then integrated over the entire length. Consider an elementary strip of breadth $\boldsymbol{b_x}$ and length $\boldsymbol{dx}$ at a distance of $\boldsymbol{x}$ from left end.

Using the principle of similar triangles the following equation for b_x can be obtained

$$b_x = b + \frac{B-b}{L}x = b + kx \text{ ,} where\ k = \frac{B-b}{L}$$

Cross–sectional area of the bar at x : $A_x = b_x t = (b+kx)t$

Axial stress at x:$\sigma_x = \frac{P}{A_x} = \frac{P}{(b+kx)t}$

Change in length over dx :$\delta dx = \frac{\sigma_x\, dx}{E} = \frac{P\, dx}{Et(b+kx)}$

Total change in length: $\delta L = \int_0^L \frac{P\, dx}{Et(b+kx)} = \frac{P}{Etk}[ln(b+kx)]_0^L$

Upon substituting the limits : $\delta L = \frac{P}{Etk}[ln(b+kL) - \ln(b)]$

$But\ (b+kL) = b + \frac{B-b}{L}L = B$

With the above substitution: $\delta L = \frac{P}{Etk}[ln(B) - \ln(b)] = \frac{P}{Etk}\ln(B/b)$

Substituting for $k = \frac{B-b}{L}$ in the above expression, following equation for elongation of tapering bar of rectangular section can be obtained

$$\delta L = \frac{P\,L}{Et(B-b)}\ln(B/b)$$

Example 5

An aluminium flat of a thickness of 8 mm and an axial length of 500 mm has a width of 15 mm tapering to 25 mm over the total length. It is subjected to an axial compressive force *P*, so that the total change in the length of flat does not exceed 0.25 mm. What is the magnitude of *P*, if *E* = 67,000 N/mm^2 for aluminium?

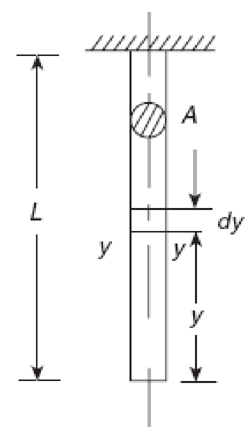
t = 8mm, B = 25mm,b = 15mm, L = 500 mm, δL = 0.25 mm, E = 67000MPa, P =?

$$P = \frac{Et(B-b)\delta L}{ln(B/b)\,L} = \frac{67000 \times 8 \times (25-15) \times 0.25}{ln(25/15) \times 500} = 5.246kN$$

Note:

Instead of using the formula, this problem can be solved from first principles as indicated in section 1.16.

Elongation in Bar Due to Self-Weight



Consider a bar of a cross-sectional area of **A** and a length **L** is suspended vertically with its upper end rigidly fixed as shown in the adjoining figure. Let the weight density of the bar is **ρ**. Consider a section y- y at a distance y from the lower end.

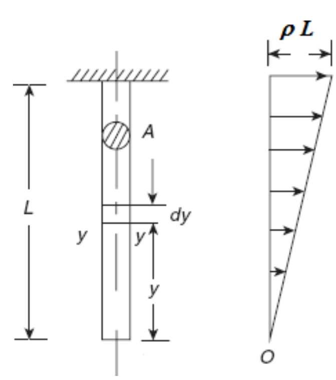
Weight of the portion of the bar below y-y = ρ A y
Stress at y-y : σ_y = ρ A y /A = ρ y
Strain at y-y : ε_y = ρ y / E
Change in length over dy: δdy = ρ y dy / E

Total change in length : $\delta L = \int_0^L \frac{\rho\, y\, dy}{E} = \left[\frac{\rho y^2}{2E}\right]_0^L = \frac{\rho L^2}{2E}$

This can also be written as : $\delta L = \frac{(\rho AL)L}{2AE} = \frac{WL}{2AE}$

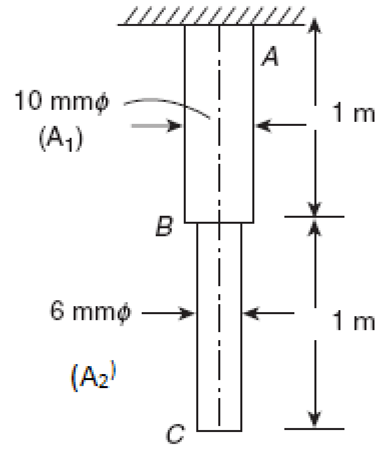
W = ρA L represents the total weight of the bar

Note:
The stress in the bar gradually increases linearly from zero at bottom to ρL at top as shown below.



Example 6

A stepped steel bar is suspended vertically. The diameter in the upper half portion is 10 mm, while the diameter in the lower half portion is 6 mm. What are the stresses due to self-weight in sections B and A as shown in the figure. E = 200 kN/mm^2. Weight density, ρ = 0.7644x10^{-3} N/mm^3. What is the change in its length if E = 200000 MPa?



Stress at B will be due to weight of portion of the bar BC

Sectional area of BC: $A_2 = \pi \times 6^2/4 = 28.27$ mm^2

Weight of portion BC: $W_2 = \rho A_2 L_2 = 0.7644\times 10^{-3} \times 28.27 \times 1000 = 21.61$N

Stress at B: $\sigma_B = W_2/A_2 = 21.61/28.27 =$ **0.764 MPa**

Stress at A will be due to weight of portion of the bar BC + AB

Sectional area of AB: $A_1 = \pi \times 10^2/4 = 78.54$ mm^2

Weight of portion AB: $W_1 = \rho A_1 L_1 = 0.7644\times 10^{-3} \times 78.54 \times 1000 = 60.04$N

Stress at A: $\sigma_c = (W_1+W_2)/A_1 = (60.04+ 21.61) / 78.54 =$ **1.04 MPa**

Change in Length in portion BC

This is caused due to weight of BC and is computed as:

$$\delta L_{BC} = \frac{W_2 L_2}{2A_2 E} = \frac{21.61\times 1000}{2\times 28.27\times 200000} = 0.00191\text{mm}$$

Change in Length in portion AB

This is caused due to weight of AB and due to weight of BC acting as a concentrated load at B and is computed as:

$$\delta L_{AB} = \frac{W_1 L_1}{2A_1 E} + \frac{W_2 L_1}{E\ A_1} = \frac{60.04\times 1000}{2\times 78.54\times 200000} + \frac{21.61\times 1000}{200000\times 78.54} = 0.0033\text{mm}$$

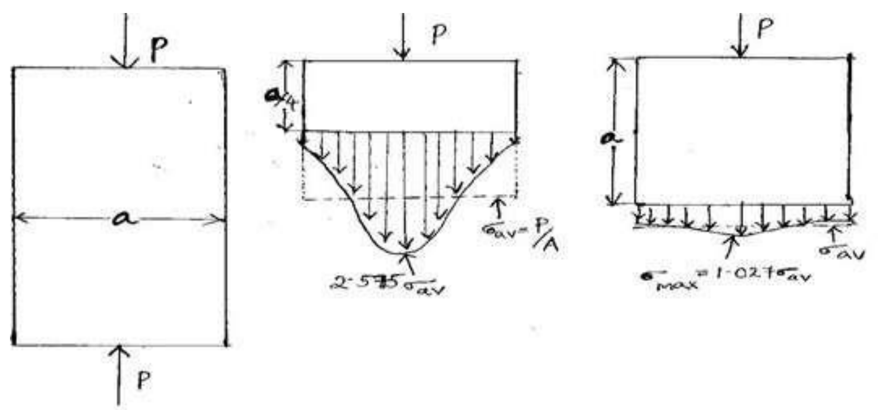
Total change in length = 0.00191+ 0.0033 = **0.00521mm**

Saint Venant's principle

In 1855, the French Elasticity theorist *Adhemar Jean Claude Barre de Saint-Venant* stated that the difference between the effects of two different but statically equivalent loads becomes very small at sufficiently large distances from the load. The stresses and strains in a body at points that are sufficiently remote from points of application of load depend only on the static resultant of the loads and not on the distribution of loads.

Stress concentration is the increase in stress along the cross-section that maybe caused by a point load or by any another discontinuity such as a hole which brings about an abrupt change in the cross sectional area.

In St.Venant"s Principle experiment, we fix two strain gages, one near the central portion of the specimen and one near the grips of the Universal Testing Machine"s (UTM) upper (stationary) holding chuck.. The respective strain values obtained from both the gages are measured and then plotted with respect to time. Since stress is proportional to strain, as per St.Venant"s principle, the stress will be concentrated near the point of application of load. Although the average stress along the uniform cross section remains constant, at the point of application of load, the stress is distributed as shown in figure below with stress being concentrated at the load point. The further the distance from the point of application of load, the more uniform the stress is distributed across the cross section.



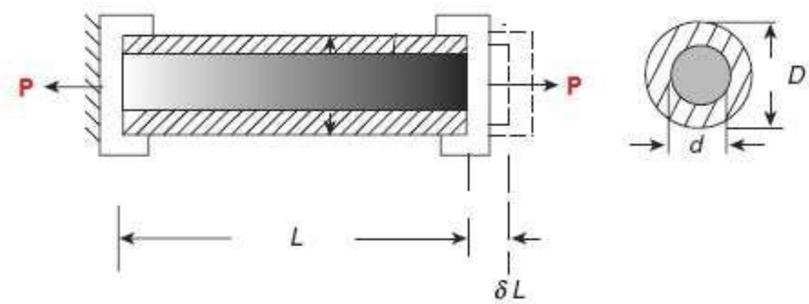
Compound or composite bars

A composite bar can be made of two bars of different materials rigidly fixed together so that both bars strain together under external load. As the strains in the two bars are same, the stresses in the two bars will be different and depend on their respective modulus of elasticity. A stiffer bar will share major part of external load.

In a composite system the two bars of different materials may act as suspenders to a third rigid bar subjected to loading. As the change in length of both bars is the same, different stresses are produced in two bars.

Stresses in a Composite Bar

Let us consider a composite bar consisting of a solid bar, of diameter ***d*** completely encased in a hollow tube of outer diameter ***D*** and inner diameter ***d*,** subjected to a tensile force **P** as shown in the following figure.



Let the extension of composite bar of length L be δL. Let E_S and E_H be the modulus of elasticity of solid bar and hollow tube respectively. Let σ_S and σ_H be the stresses developed in the solid bar and hollow tube respectively.

Since change in length of solid bar is equal to the change in length of hollow tube, we can establish the relation between the stresses in solid bar and hollow tube as shown below :

$$\frac{\sigma_S L}{E_S} = \frac{\sigma_H L}{E_H} \text{ or } \sigma_S = \sigma_H \frac{E_S}{E_H}$$

Area of cross section of the hollow tube : $A_H = \frac{\pi(D^2 - d^2)}{4}$

Area of cross section of the solid bar : $A_S = \frac{\pi d^2}{4}$

Load carried by the hollow tube : $P_H = \sigma_H A_H$ and Load carried by the solid bar : $P_S = \sigma_S A_S$

But $P = P_S + P_H = \sigma_S A_S + \sigma_H A_H$

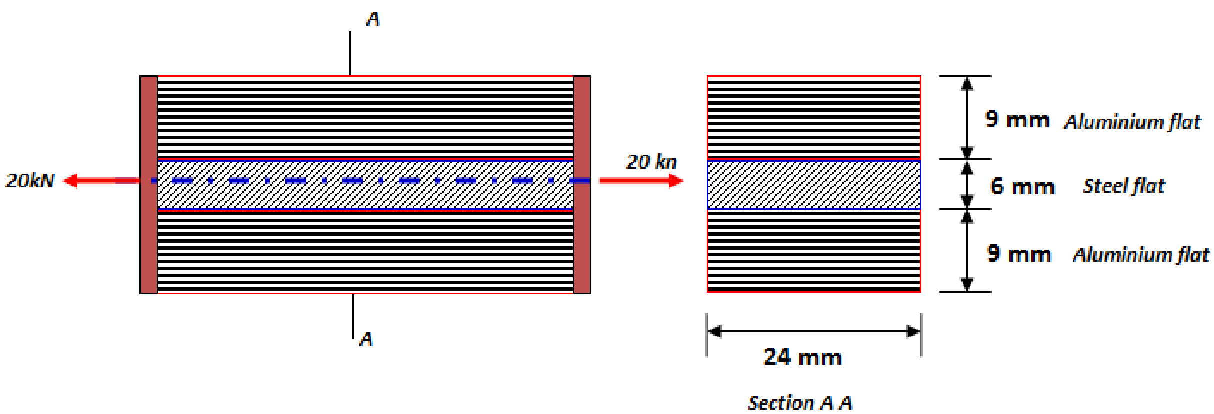
With $\sigma_S = \sigma_H \frac{E_S}{E_H}$, the following equation can be written

$$P = \sigma_H \frac{E_S}{E_H} A_s + \sigma_H A_H = \sigma_H \left(A_H + \frac{E_S}{E_H} A_s\right)$$

E_S/E_H is called *modular ratio*. Using the above equation stress in the hollow tube can be calculated. Next, the stress in the solid bar can be calculated using the equation $P = \sigma_S A_S + \sigma_H A_H$.

Example 7

A flat bar of steel of 24 mm wide and 6 mm thick is placed between two aluminium alloy flats 24 mm × 9 mm each. The three flats are fastened together at their ends. An axial tensile load of 20 kN is applied to the composite bar. What are the stresses developed in steel and aluminium alloy? Assume E_S = 210000 MPa and E_A = 70000MPa.



Area of Steel flat: A_S = 24 x 6 = 144 mm^2

Area of Aluminium alloy flats: A_A = 2 x 24 x 9 = 432 mm^2

Since all the flats elongate by the same extent, we have the condition that $\frac{\sigma_S L}{E_S} = \frac{\sigma_A L}{E_A}$.

The relationship between the stresses in steel and aluminum flats can be established as:

$$\sigma_S = \sigma_A \frac{E_S}{E_A} = 3\,\sigma_A$$

Since $P = P_S + P_A = \sigma_S A_S + \sigma_A A_A$. This can be written as

$$P = 3\sigma_A A_s + \sigma_A A_A = \sigma_A (3A_s + A_A)$$

From which stress in aluminium alloy flat can be computed as:

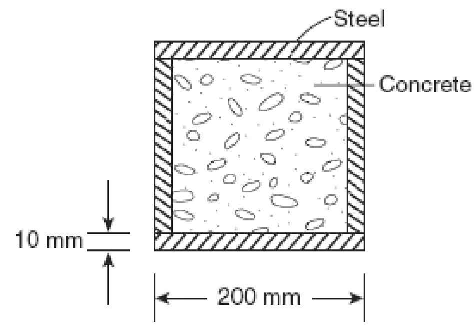
$$\sigma_A = \frac{P}{(3A_s + A_A)} = \frac{20 \times 1000}{(3 \times 144 + 432)} = \mathbf{23.15\,MPa}$$

Stress in steel flat can be computed as:

$$\sigma_S = 3 \times 23.15 = \mathbf{69.45\,MPa}$$

Example 8

A short post is made by welding steel plates into a square section and then filling inside with concrete. The side of square is 200 mm and the thickness t = 10 mm as shown in the figure. The steel has an allowable stress of 140 N/mm^2 and the concrete has an allowable stress of 12 N/mm^2. Determine the allowable safe compressive load on the post. E_C = 20 GPa, Es = 200 GPa.



Since the composite post is subjected to compressive load, both concrete and steel tube will shorten by the same extent. Using this condition following relation between stresses in concrete and steel can be established.

$$\frac{\sigma_C L}{E_C} = \frac{\sigma_S L}{E_S} \text{ or } \sigma_S = \sigma_C \frac{E_S}{E_C} = 10\,\sigma_C$$

Assume that load is such that σ_s = 140 N/mm^2. Using the above relationship, the stress in concrete corresponding to this load can be calculated as follows:

$$140 = 10\,\sigma_C \;\; or \;\; \sigma_C = 14\, N/mm^2 > 12 \text{ N/mm}^2$$

Hence the assumed load is not a safe load.

Instead assume that load is such that σ_c = 12 N/mm^2. The stress in steel corresponding to this load can be calculated as follows:

$$\sigma_s = 12 \times 10 \;\; or \;\; \sigma_s = 120\, N/mm^2 < 140 \text{ N/mm}^2$$

Hence the assumed load is a safe load which is calculated as shown below.

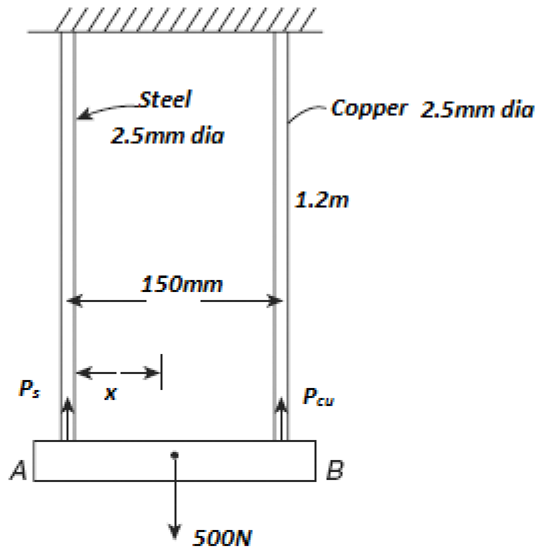
Area of concrete section Ac = 180 x180 = 32400mm^2.

Area of steel tube As = 200 x 200 – 32400 = 7600 mm^2.

$$P = \sigma_C A_C + \sigma_s A_s = 12 \times 32400 + 120 \times 7600 = \mathbf{1300.8}\boldsymbol{kN}$$

Example 9

A rigid bar is suspended from two wires, one of steel and other of copper, length of the wire is 1.2 m and diameter of each is 2.5 mm. A load of 500 N is suspended on the rigid bar such that the rigid bar remains horizontal. If the distance between the wires is 150 mm, determine the location of line of application of load. What are the stresses in each wire and by how much distance the rigid bar comes down? Given $E_s = 3E_{cu} = 201000$ N/mm^2.



i. Area of copper wire (Acu) = Area of steel wire(As) = π x 2.5^2/4 = 4.91 mm^2

ii. For the rigid bar to be horizontal, elongation of both the wires must be same. This condition leads to the following relationship between stresses in steel and copper wires as:

$$\sigma_s = \frac{E_s}{E_{cu}}\sigma_{cu} = 3\sigma_{cu}$$

iii. Using force equilibrium, the stress in copper and steel wire can be calculated as:

$$P = P_s + P_{cu} = \sigma_s A_s + \sigma_{cu} A_{cu} = 3\,\sigma_{cu} A_s + \sigma_{cu} A_{cu} = \sigma_{cu}(3A_s + A_{cu})$$

$$\sigma_{cu} = \frac{P}{(A_{cu} + 3A_s)} = \frac{500}{(4.91 + 3 \times 4.91)} = 25.46\ MPa$$

$$\sigma_s = 3 \times 25.46 = 76.37\ MPa$$

iv. Downward movement of rigid bar = elongation of wires

$$\delta L_s = \frac{\sigma_s}{E_s}L = \frac{76.37}{201000} \times 1200 = 0.456\ mm$$

v. Position of load on the rigid bar is computed by equating moments of forces carried by steel and copper wires about the point of application of load on the rigid bar.

$$P_s\, x = P_c\,(150 - x)$$
$$(76.37 \times 4.91)x = (25.46 \times 4.91)\,(150 - x)$$
$$\frac{x}{150 - x} = 0.333$$

$$x = \mathbf{37.47mm}\ from\ steel\ wire$$

Note:
If the load is suspended at the centre of rigid bar, then both steel and copper wire carry the same load. Hence the stress in the wires is also same. As the moduli of elasticity of wires are different, strains in the wires will be different. This results in unequal elongation of wires causing the rigid bar to rotate by some magnitude. This can be prevented by offsetting the load or with wires having different length or with different diameter such that elongation of wires will be same.

Example 10

A load of 2MN is applied on a column 500mm x 500mm. The column is reinforced with four steel bars of 12mm dia, one in each corner. Find the stresses in concrete and steel bar. Es = 2.1 $\times 10^5$ N/mm^2 and Ec = 1.4 x 10^4 N/mm^2.

i. Area of steel bars: As= 4 x (π x $12^2/4$) = 452.4 mm^2

ii. Area of concrete: Ac = 500 x500 – 452.4 = 249547.6 mm^2

iii. Relation between stress in steel and concrete : $\sigma_s = \frac{E_S}{E_c}\sigma_c = \frac{2.1\times10^5}{1.4\times10^4}\sigma_c = 15\sigma_c$

iv. $P = P_s + P_c = \sigma_s A_s + \sigma_c A_c = 15\,\sigma_c A_s + \sigma_c A_c = \sigma_c(15A_s + A_c)$

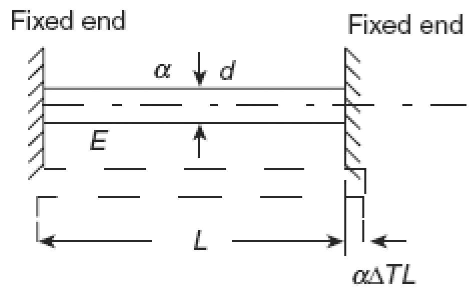
v. $Stress\ in\ concrete\ \sigma_c = \frac{P}{(A_c + 15A_S)} = \frac{2\times10^6}{(249547.6 + 15\times452.4)} = \mathbf{7.8\ MPa}$

vi. $Stress\ in\ steel\ \sigma_s = 15\sigma_c = 15\times7.8 = \mathbf{117 MPa}$

Temperature stresses in a single bar

If a bar is held between two unyielding (rigid) supports and its temperature is raised, then a compressive stress is developed in the bar as its free thermal expansion is prevented by the rigid supports. Similarly, if its temperature is reduced, then a tensile stress is developed in the bar as its free thermal contraction is prevented by the rigid supports. Let us consider a bar of diameter ***d*** and length ***L*** rigidly held between two supports as shown in the following figure. Let

$\boldsymbol{\alpha}$ be the coefficient of linear expansion of the bar and its temperature is raised by $\Delta \boldsymbol{T}$ (°C)



- Free thermal expansion in the bar = $\alpha\,\Delta T\,L$.
- Since the supports are rigid, the final length of the bar does not change. The fixed ends exert compressive force on the bar so as to cause shortening of the bar by $\alpha\,\Delta T\,L$.
- Hence the compressive strain in the bar = $\alpha\,\Delta T\,L\,/\,L = \alpha\,\Delta T$
- Compressive stress = $\alpha\,\Delta T\,E$
- Hence the thermal stresses introduced in the bar = $\alpha\,\Delta T\,E$

Note:

The bar can buckle due to large compressive forces generated in the bar due to temperature increase or may fracture due to large tensile forces generated due to temperature decrease.

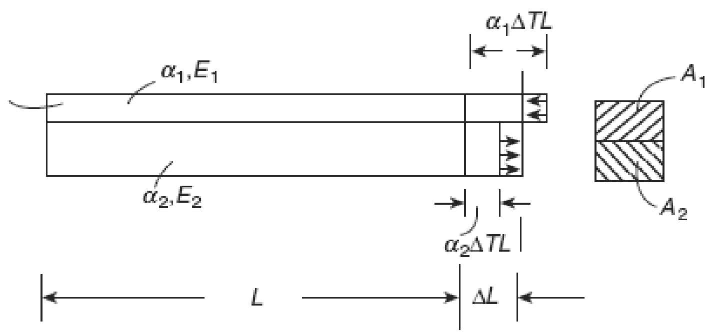
Example 11

A rail line is laid at an ambient temperature of 30°C. The rails are 30 m long and there is a clearance of 5 mm between the rails. If the temperature of the rail rises to 60°C, what is the stress developed in the rails?. Assume $\alpha = 11.5 \times 10^{-}6$/°C, E = *2,10,000 N/mm^2*

- L = 30,000 mm, $\alpha = 11.5 \times 10^{-6}$/°C, Temperature rise ΔT = 60-30 = 30°C
- Free expansion of rails = $\alpha\,\Delta T\,L = 11.5 \times 10^{-6} \times 30 \times 30000 = 10.35$mm
- Thermal expansion prevented by rails = Free expansion – clearance = 10.35 – 5 = 5.35mm
- Strain in the rails ε = 5.35/30000 = 0.000178
- Compressive stress in the rails = ε x E = 0.000178 x 210000 =**37.45N/mm^2**.

Temperature Stresses in a Composite Bar

A composite bar is made up of two bars of different materials perfectly joined together so that during temperature change both the bars expand or contract by the same amount. Since the coefficient of expansion of the two bars is different thermal stresses are developed in both the bars. Consider a composite bar of different materials with coefficients of expansion and modulus of elasticity, as α_1, E_1 and α_2, E_2, respectively, as shown in the following figure. Let the temperature of the bar is raised by ΔT *and* $\alpha_1 > \alpha_2$



Free expansion in bar 1 = $\alpha_1 \Delta T\, L$ and Free expansion in bar 2 = $\alpha_2\, \Delta T\, L$. Since both the bars expand by ΔL together we have the following conditions:

- Bar 1: $\Delta L < \alpha_1\, \Delta T\, L$. The bar gets compressed resulting in compressive stress
- Bar 2: $\Delta L > \alpha_2 \Delta T\, L$. The bar gets stretched resulting in tensile stress.

Compressive strain in Bar 1 : $\varepsilon_1 = \frac{\alpha_1 \Delta T L - \Delta L}{L}$

Tensile strain in Bar 2 : $\varepsilon_2 = \frac{\Delta L - \alpha_2 \Delta T L}{L}$

$$\varepsilon_1 + \varepsilon_2 = \frac{\alpha_1 \Delta T L - \Delta L}{L} + \frac{\Delta L - \alpha_2 \Delta T L}{L} = (\alpha_1 - \alpha_2)\Delta T$$

Let σ_1 and σ_2 be the temperature stresses in bars. The above equation can be written as:

$$\frac{\sigma_1}{E_1} + \frac{\sigma_2}{E_2} = (\alpha_1 - \alpha_2)\Delta T$$

In the absence of external forces, for equilibrium, compressive force in Bar 1 = Tensile force in Bar 2. This condition leads to the following relation

$$\sigma_1 A_1 = \sigma_2 A_2$$

Using the above two equations, temperature stresses in both the bars can be computed. This is illustrated in the following example.

Note:

If the temperature of the composite bar is reduced, then a tensile stress will be developed in bar 1 and a compressive stress will be developed in bar 2 , since $\alpha_1 > \alpha_2$.

Example 12

A steel flat of 20 mm × 10 mm is fixed with aluminium flat of 20 mm × 10 mm so as to make a square section of 20 mm × 20 mm. The two bars are fastened together at their ends at a temperature of 26°C. Now the temperature of whole assembly is raised to 55°C. Find the stress in each bar. E_s = 200 GPa, E_a = 70 GPa, $\alpha_s = 11.6 \times 10^{-6}$/°C, $\alpha_a = 23.2 \times 10^{-6}$/°C.

- Net temperature rise, ΔT = 55 − 26 = 29°C.
- Area of Steel flat (As) = Area of Aluminium flat (Aa) = 20 x10 =200 mm2
- For equilibrium, $\sigma_s A_s = \sigma_a A_a$; $\sigma_s = \sigma_a$ will be one of the conditions to be satisfied by the composite assembly.
- $But\ \frac{\sigma_a}{E_a} + \frac{\sigma_s}{E_s} = (\alpha_a - \alpha_s)\Delta T = (23.2 - 11.6) \times 29 \times 10^{-6} = 0.000336$
- $\frac{\sigma_s}{200000} + \frac{\sigma_a}{70000} = 0.000336$
- $270000\ \sigma_s = 4709600$;
- $\sigma_s(tensile) = \sigma_a(compressive) = 17.44MPa$ as $\alpha_a > \alpha_s$

Example 13

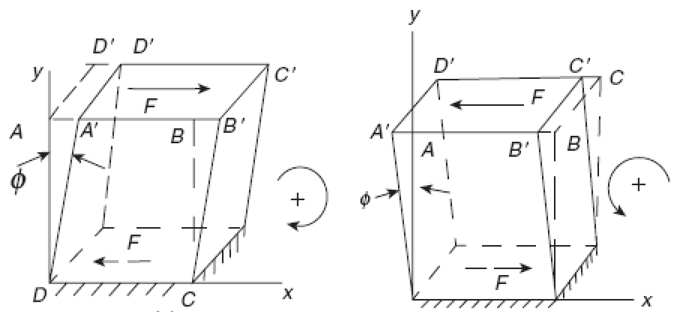
A flat steel bar of 20 mm × 8 mm is placed between two copper bars of 20 mm × 6 mm each so as to form a composite bar of section of 20 mm × 20 mm. The three bars are fastened together at their ends when the temperature of each is 30°C. Now the temperature of the whole assembly is raised by 30°C. Determine the temperature stress in the steel and copper bars. $E_s = 2E_{cu}$= 210 kN/mm^2, $\alpha_s = 11 \times 10^{-6}$/°C, $\alpha_{cu} = 18 \times 10^{-6}$/°C.

- Net temperature rise, ΔT = 30°C.
- Area of Steel flat (A_s) = 20 x 8 = 160 mm^2
- Area of Copper flats (A_{cu}) = 2 x 20 x 6 =240 mm^2

- For equilibrium, $\sigma_s A_s = \sigma_{cu} A_{cu}$; $\sigma_s = 1.5\,\sigma_{cu}$ will be one of the conditions to be satisfied by the composite assembly.
- $But\ \frac{\sigma_{cu}}{E_{cu}} + \frac{\sigma_s}{E_s} = (\alpha_{cu} - \alpha_s)\Delta T = (18 - 11) \times 30 \times 10^{-6} = 0.00021$
- $\frac{\sigma_{cu}}{105000} + \frac{1.5\sigma_{cu}}{210000} = 0.00021$
- $\sigma_{cu} = 12.6$MPa (compressive) and $\sigma_s = 18.9$MPa (tensile) as $\alpha_{cu} > \alpha_s$

Simple Shear stress and Shear Strain

Consider a rectangular block which is fixed at the bottom and a force *F* is applied on the top surface as shown in the figure (a) below.



(a) (b)

Equal and opposite reaction *F* develops on the bottom plane and constitutes a couple, *tending to rotate the body in a clockwise direction*. This type of shear force is a *positive shear force* and the shear force per unit surface area on which it acts is called *positive shear stress* (τ). If force is applied in the opposite direction as shown in Figure (b), then they are termed as negative shear force and shear stress.

The *Shear Strain* (ϕ) = AA''/AD = tanϕ. Since ϕ is a very small quantity, tanϕ ≈ ϕ. Within the elastic limit, $\tau \propto \phi$ *or* $\tau = G\,\phi$

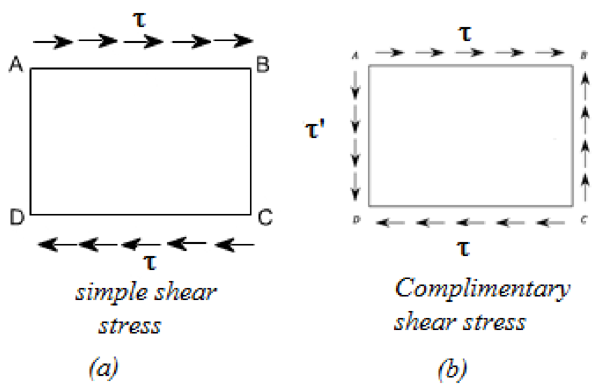
The constant of proportionality **G** is called *rigidity modulus or shear modulus.*

Note:

Normal stress is computed based on area perpendicular to the surface on which the force is acting, while, the shear stress is computed based on the surface area on which the force is acting. Hence shear stress is also called tangential stress.

Complementary Shear Stresses

Consider an element ABCD subjected to shear stress (τ) as shown in figure (a). We cannot have equilibrium with merely equal and opposite tangential forces on the faces AB and CD as these forces constitute a couple and induce a turning moment. The statical equilibrium demands that there must be tangential components (τ'') along AD and CB such that that can balance the turning moment. These tangential stresse (τ'') is termed as *complimentary shear stress*.



Let t be the thickness of the block. Turning moment due to τ will be $(\tau \times t \times L_{AB})\, L_{BC}$ and Turning moment due to τ' will be $(\tau' \times t \times L_{BC})\, L_{AB}$. Since these moments have to be equal for equilibrium we have:

$$(\tau \times t \times L_{AB})\, L_{BC} = (\tau' \times t \times L_{BC})\, L_{AB}.$$

From which it follows that $\tau = \tau'$, that is, intensities of shearing stresses across two mutually perpendicular planes are equal.

Volumetric strain

This refers to the slight change in the volume of the body resulting from three mutually perpendicular and equal direct stresses as in the case of a body immersed in a liquid under pressure. This is defined as the *ratio of change in volume to the original volume* of the body.

Consider a cube of side '**a**' strained so that each side becomes '**a ± δa'.**

- Hence the linear strain = δa/a.
- Change in volume = $(a \pm \delta a)^3 - a^3 = \pm 3a^2\delta a$. (ignoring small higher order terms)
- Volumetric strain $\varepsilon_v = \pm 3a^2\, \delta a/a^3 = \pm 3\,\delta a/a$
- *The volumetric strain is three times the linear strain*

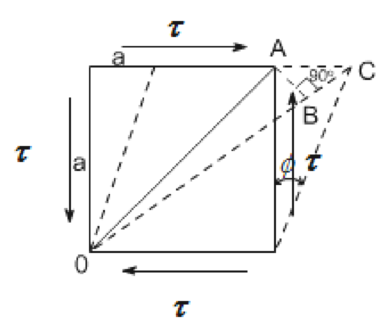
Bulk Modulus

This is defined as the ratio of the normal stresses (p) to the volumetric strain (ε_v) and denoted by **'K'.** Hence $\mathbf{K = p/\varepsilon_v}$. This is also an elastic constant of the material in addition to E, G and ν.

Relation between elastic constants

Relation between E,G and ν

Consider a cube of material of side „a' subjected to the action of the shear and complementary shear stresses and producing the deformed shape as shown in the figure below.

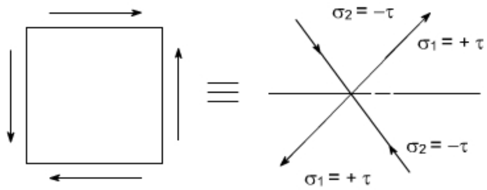


- Since, within elastic limits, the strains are small and the angle ACB may be taken as 45^0.
- Since angle between OA and OB is very small hence OA ≈ OB. BC, is the change in the length of the diagonal OA
- Strain on the diagonal OA = Change in length / original length = BC/OA

$$= AC \cos 45/ (a/\sin 45) = AC/ 2a = a\,\phi / 2\,a = \phi / 2$$

- It is found that *strain along the diagonal is numerically half the amount of shear stain*.
- But from definition of rigidity modulus we have, $G = \tau/\phi$
- Hence, Strain on the diagonal OA = $\tau / 2G$

The shear stress system is equivalent or can be replaced by a system of direct stresses at 45^0 as shown below. One set will be compressive, the other tensile, and both will be equal in value to the applied shear stress.



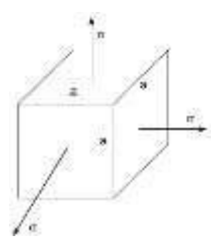
Strain in diagonal OA due to direct stresses = $\frac{\sigma_1}{E} - \nu\frac{\sigma_2}{E} = \frac{\tau}{E} + \nu\frac{\tau}{E} = \frac{\tau}{E}(1+\nu)$

Equating the strain in diagonal OA we have $\frac{\tau}{2G} = \frac{\tau}{E}(1+\nu)$

Relation between E,G and ν can be expressed as : $E = 2G(1+\nu)$

Relation between E,K and ν

Consider a cube subjected to three equal stresses a shown in the figure below.



Strain in any one direction = $\frac{\sigma}{E} - \nu\frac{\sigma}{E} - \nu\frac{\sigma}{E} = \frac{\sigma}{E}(1-2\nu)$

Since the volumetric strain is three times the linear strain: $\varepsilon_v = 3\frac{\sigma}{E}(1-2\nu)$

From definition of bulk modulus : $\varepsilon_v = \frac{\sigma}{K}$

$$3\frac{\sigma}{E}(1-2\nu) = \frac{\sigma}{K}$$

Relation between E,K and ν can be expressed as : $E = 3K(1-2\nu)$

Note: Theoretically $\nu < 0.5$ as E cannot be zero

Relation between E, G and K

We have E = 2G(1+ν) from which ν = (E - 2G) / 2G
We have E = 3K(1-2ν) from which ν = (3K -E) / 6K

(E - 2G) / 2G = (3K -E) / 6K or (6EK - 12GK) = (6GK - 2EG) or 6EK+2EG = (6GK +12GK)

Relation between E,G and K can be expressed as: $E = \frac{9GK}{(3K+G)}$

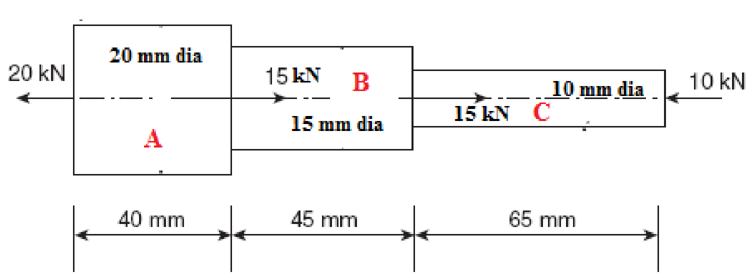
Exercise problems

1. A steel bar of a diameter of 20 mm and a length of 400 mm is subjected to a tensile force of 40 kN. Determine (a) the tensile stress and (b) the axial strain developed in the bar if the Young‟s modulus of steel E = 200 kN/mm^2
   Answer: (a) Tensile stress = 127.23MPa, (b) Axial strain = 0.00064

2. A 100 mm long bar is subjected to a compressive force such that the stress developed in the bar is 50 MPa. (a) If the diameter of the bar is 15 mm, what is the axial compressive force? (b) If E for bar is 105 kN/mm^2, what is the axial strain in the bar?
   Answer: (a) Compressive force = 8.835 kN, (b) Axial strain = 0.00048

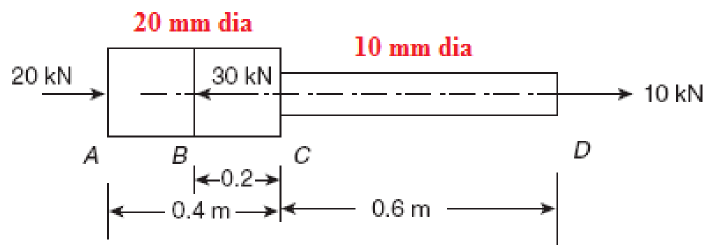
3. A steel bar of square section 30 × 30 mm and a length of 600 mm is subjected to an axial tensile force of 135 kN. Determine the changes in dimensions of the bar. E = 200 kN/mm^2, ν = 0.3.
   Answer: Increase in length δl = 0.45 mm, Decrease in breadth δb = 6.75 × 10^{-3} mm,

4. A stepped circular steel bar of a length of 150 mm with diameters 20, 15 and 10 mm along lengths 40, 50 and 65 mm, respectively, subjected to various forces is shown in figure below. If E = 200 kN/mm^2, determine the total change in its length.



Answer : Total decrease in length = 0.022mm

5. A stepped bar is subjected to axial loads as shown in the figure below. If *E* = 200 GPa, calculate the stresses in each portion *AB*, *BC* and *CD*. What is the total change in length of the bar?

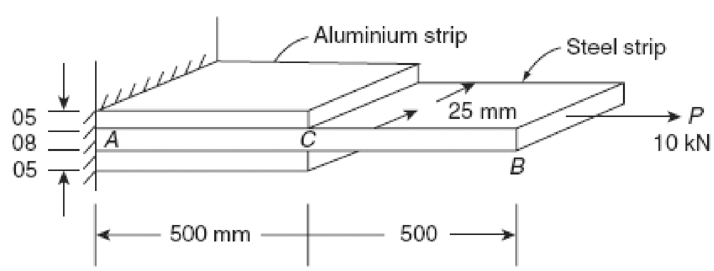


Answer: Total increase in length = 0.35mm

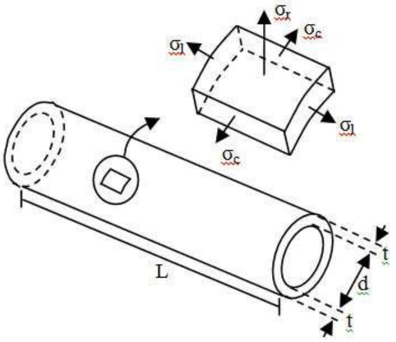
6. A 400-mm-long aluminium bar uniformly tapers from a diameter of 25 mm to a diameter of 15 mm. It is subjected to an axial tensile load such that stress at middle section is 60 MPa. What is the load applied and what is the total change in the length of the bar if E = 67,000 MPa? (*Hint: At the middle diameter = (25+15)/2 = 20 mm*).
Answer: Load = 18.85kN, Increase in length = 0.382 mm

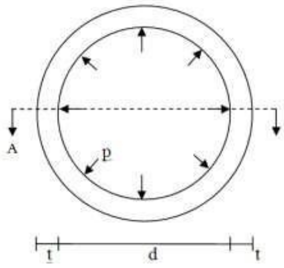
7. A short concrete column of 250 mm × 250 mm in section strengthened by four steel bars near the corners of the cross-section. The diameter of each steel bar is 30 mm. The column is subjected to an axial compressive load of 250 kN. Find the stresses in the steel and the concrete. Es = 15 Ec = 210 GPa. If the stress in the concrete is not to exceed 2.1 N/mm^2, what area of the steel bar is required in order that the column may support a load of 350 kN?
Answer: Stress in concrete = 2.45N/mm^2, Stress in steel = 36.75N/mm^2, Area of steel = 7440 mm^2

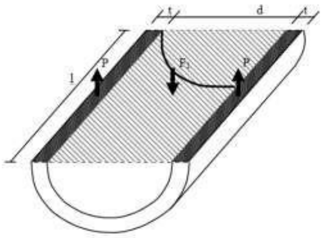
8. Two aluminium strips are rigidly fixed to a steel strip of section 25 mm × 8 mm and 1 m long. The aluminium strips are 0.5 m long each with section 25 mm × 5 mm. The composite bar is subjected to a tensile force of 10 kN as shown in the figure below. Determine the deformation of point B. Es = 3EA = 210 kN/mm^2. *Answer: 0.203mm*
(*Hint: Portion CB is a single bar, Portion AC is a composite bar. Compute elongation separately for both the portions and add*)

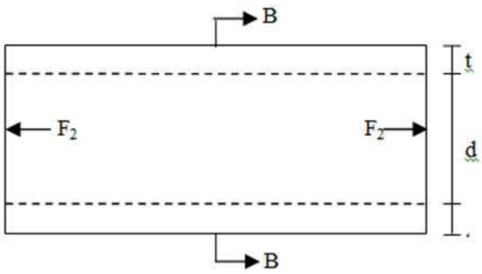


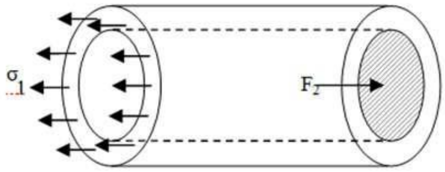
Module-2



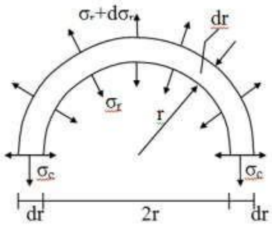




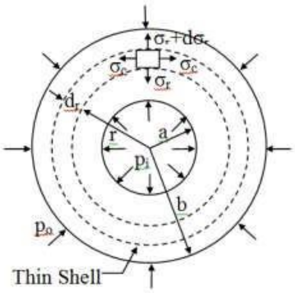




SECTION B – B



Free body diagram



Introduction, Thin cylinders subjected to internal pressure; Hoop stresses, Longitudinal stress and change in volume. Thick cylinders subjected to both internal and external pressure; Lame's equation, radial and hoop stress distribution.

Cylinders are pressure vessels such as pipes, steam boilers, storage tanks, etc., which carry gas or fluid under pressure. A cylinder is said to be thin if the thickness $\leq \frac{1}{10}$ internal diameter and thick if the thickness $> \frac{1}{10}$ internal diameter.

3.2 TYPES OF STRESSES IN CYLINDERS

Consider a thin cylinder subjected to an internal pressure 'p'. The walls of the cylinders are generally subjected to three types of normal stresses which are discussed below. The enlarged view of a portion of the wall on which the three stresses are acting is shown in Fig.1.

3.2.1 Circumferential Stress (σ_c)

It is the normal stress which acts along the circumference of the cylinder (Fig.1). It is also called as hoop stress or girth stress. It is denoted as σ_c

3.2.2 Longitudinal Stress (σ_l)

It is the normal stress which acts along the length of the cylinder (Fig.1). It is denoted as σ_l

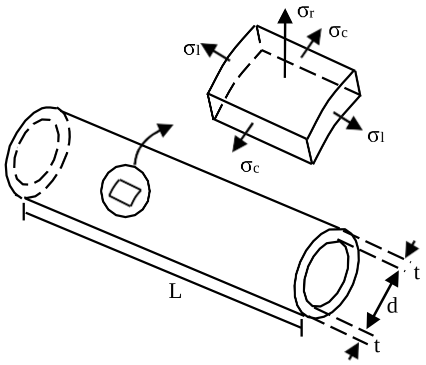


Fig. 1 Wall of thin cylinder subjected to three stresses σ_c, σ_l and σ_r.

3.2.3 Radial Stress (σ_r)

It is the normal stress which acts along the radial direction (Fig.1). It is denoted as σ_r

3.3 <u>THIN CYLINDER THEORY</u>

3.3.1 Assumptions

The assumptions made in the thin cylinder theory are;

- The magnitude of radial stress being very small is neglected.
- The distribution of circumferential stress across the cross-section is assumed to be uniform since the thickness of the cylinder wall is very small.

3.3.2 Circumferential and Longitudinal Stresses in Thin Cylinders

Consider a thin cylinder of internal diameter 'd', thickness 't' and length 'l' subjected to an internal pressure 'p' as shown in Fig. 2.

Circumferential Stress (σ_c)

Consider the longitudinal section A – A through the cylinder as shown in Fig.2. The free body diagram of the lower-half portion of the cylinder is shown in Fig. 3.

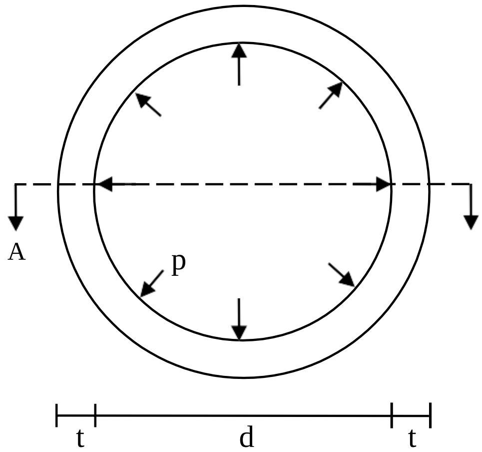


Fig. 2 Thin cylinder of internal diameter '*d*', thickness '*t*' and length '*l*' subjected to an internal pressure '*p*'

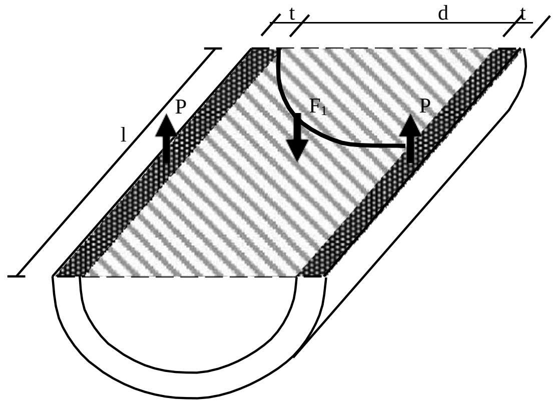


Fig. 3 Free body diagram of the lower-half portion of the cylinder

It is apparent that the total bursting force 'F_1' (due to internal pressure p) acting normal to the cutting plane A - A, is resisted by equal forces P acting on each cut surface of the cylinder wall. Applying the equilibrium condition,

$$\Sigma V = 0 \; [\uparrow +ve]$$

$$-F_1 + 2P = 0 \qquad (1)$$

But $F_1 = (p)(d\,l)$ and $P = (\sigma_c\, t\, l)$

Substituting in eq. (1) $-p\,d\,l + 2\,[\sigma_c\, t\, l] = 0$

$$\sigma_c = \frac{p\,d}{2\,t} \qquad (2)$$

Longitudinal Stress 'σ_l'

Consider a thin cylinder closed at the ends subjected to an internal pressure 'p'. Take a transverse section B-B as shown in Fig. 4. The free body diagram of cut portion of the thin cylinder to the right of transverse section B-B is also shown in Fig. 4. It is apparent that the total bursting force 'F_2' (due to internal pressure p) is resisted by normal stress σ_l developed on the cylinder wall at the cut surface B-B.

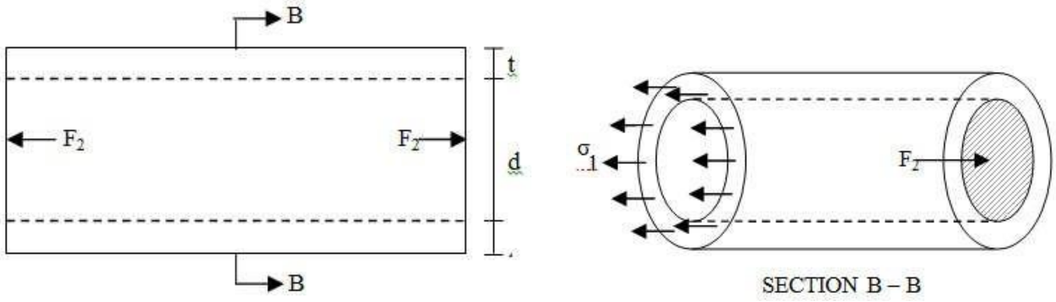


Fig. 4 Closed thin cylinder showing bursting force F2 and Free body diagram of the cylinder towards right of B-B

Applying the equilibrium condition,

$$\Sigma H = 0 \; [\rightarrow + ve]$$

$$F_2 - \sigma_l \,(A) = 0 \qquad (3)$$

But $\quad F_2 = p\left(\frac{\pi}{4} d^2\right)$

For thin cylinders, the cross sectional area can be approximated as

$$A = \text{Perimeter x thickness} = (\pi d)\, t$$

Substituting in (3)

$$p\left(\frac{\pi}{4} d^2\right) - \sigma_l(\pi d t) = 0$$

Hence $\qquad \sigma_l = \dfrac{pd}{4t} \qquad (4)$

Comparing (2) and (4) $\qquad \sigma_c = 2\,\sigma_l \qquad (5)$

Circumferential stress = 2 x Longitudinal stress

3.3.3 Maximum Shearing Stress ($\sigma_{s\ max}$)

The only stresses that act on the walls of a thin cylinder are the circumferential stress σ_c and the longitudinal stress σ_l which are normal stresses (Fig. 5). Since the element is free of shear stress, the above stresses are themselves the principal stresses.

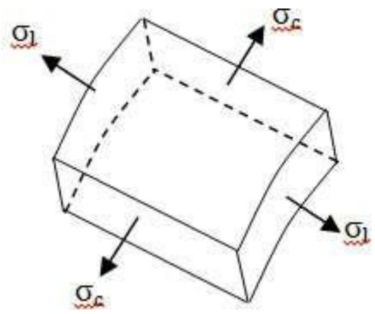


Fig. 5 Normal stresses on the wall of thin

Therefore, $$\sigma_{s\ max} = \frac{\sigma_{n1} - \sigma_{n2}}{2} \qquad (6)$$

where σ_{n1} = maximum principal stress

σ_{n2} = minimum principal stress

Here, $$\sigma_{n1} = \sigma_c = \frac{pd}{2t} \quad \text{and} \quad \sigma_{n2} = \sigma_l = \frac{pd}{4t} \qquad (7)$$

Substituting eq. (7) in eq. (6), and simplifying

$$\sigma_{s\ max} = \frac{p\ d}{8\ t} \qquad (8)$$

Note: On any plane, if shear stress is absent, normal stress acting on the plane is called principal stress.

3.3.4 Expressions for Changes in Diameter, Length and Volume

The two principal stresses which are acting at any point in the wall of a thin cylinder shell are, $\sigma_{n1} = \sigma_c$ = circumferential stress and $\sigma_{n2} = \sigma_l$ = longitudinal stress. Let ε_c, ε_l, E and υ represent the circumferential strain, longitudinal strain, Young's modulus and Poisson's ratio respectively.

Change in diameter (δd)

The circumferential strain, ε_c can be expressed in terms of circumferential stress, σ_c and longitudinal stress, σ_l as

$$\varepsilon_c = \frac{\sigma_c}{E} - \upsilon\left(\frac{\sigma_l}{E}\right) \qquad (9)$$

Substituting $\sigma_c = \frac{pd}{2t}$ and $\sigma_l = \frac{pd}{4t}$ in eq. (9), and simplifying

$$\varepsilon_c = \frac{Pd}{2t\,E}\left(1 - \frac{\upsilon}{2}\right) \qquad (10)$$

Since the circumference is directly proportional to the diameter, the strain in eq. (10) can be equated to diametral strain, ie, $\frac{\delta d}{d}$

Thus, $\varepsilon_c = \frac{\delta d}{d}$

Therefore, change in diameter $\delta d = \varepsilon_c \cdot d$ (11)

where the circumferential strain, ε_c is given in eq. (10).

Change in length (δl)

The longitudinal strain, ε_l can be expressed in terms of longitudinal stress, σ_l and circumferential stress, σ_c as

$$\varepsilon_l = \frac{\sigma_l}{E} - \upsilon \left(\frac{\sigma_c}{E}\right) \quad (12)$$

Substituting $\sigma_c = \frac{pd}{2t}$ and $\sigma_l = \frac{pd}{4t}$ in eq. (12), and simplifying

$$\varepsilon_l = \frac{pd}{2tE}\left(\frac{1}{2} - \upsilon\right) \quad (13)$$

Further $\varepsilon_l = \frac{\delta l}{l}$

Hence, change in length $\delta l = \varepsilon_l \cdot l$ (14)

where the longitudinal strain, ε_l is given in eq. (13).

Change in volume (δV)

Let 'V' be the internal volume of the cylinder

Hence, $V = \frac{\pi}{4} d^2 l$

Taking logarithms $\log V = \log \frac{\pi}{4} + 2 \log d + \log l$

Taking differentials $\frac{\delta V}{V} = 2 \frac{\delta d}{d} + \frac{\delta l}{l}$ (15)

Substituting $\frac{\delta V}{V} = \varepsilon_v$, $\frac{\delta d}{d} = \varepsilon_c$ and $\frac{\delta l}{l} = \varepsilon_l$ in eq. (15)

$$\varepsilon_v = 2\varepsilon_c + \varepsilon_l \quad (16)$$

Substituting for 'ε_c' and 'ε_l' from eqs. (10) and (13) in eq. (16)

$$\varepsilon_v = \frac{pd}{2t\,E}\left(\frac{5}{2} - 2\upsilon\right) \qquad (17)$$

Since, $$\varepsilon_v = \frac{\delta V}{V}$$

Change in volume $$\delta V = \varepsilon_v \,.V \qquad (18)$$

where the volumetric strain, ε_v is given in eq. (17).

3.3.5 Efficiency of Joints

Cylinders are normally made of number of sheets which are riveted or welded together. The joints between the sheets can be along the longitudinal direction and/or along the circumferential direction. The longitudinal and circumferential joints in a thin cylinder are shown in Fig. 6.

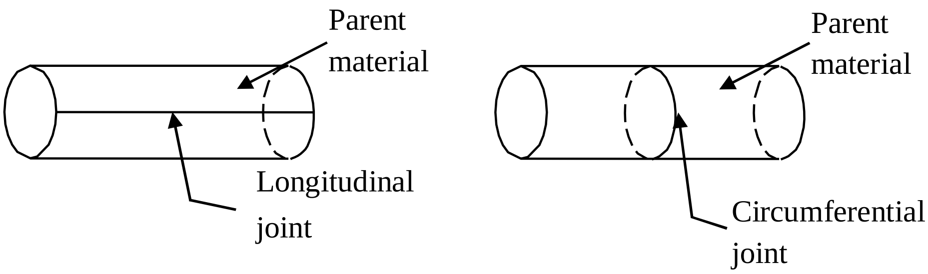


Fig. 6 Longitudinal and circumferential joints in thin cylinder

Joints are generally weaker than parent material. The ratio of strength of joint to strength of parent material is called efficiency (η) of the joint. The efficiencies of longitudinal and circumferential joints are designated as η_l and η_c respectively. A longitudinal joint resists circumferential stress 'σ_c' and circumferential joint resists longitudinal stress 'σ_l'.

- If $\eta_l \leq 2\ \eta_c$, then the longitudinal joint becomes critical and hence the following expression governs the design

$$\sigma_c\,\eta_l = \frac{p\,d}{2t} \qquad (19)$$

where σ is equated to the safe or permissible stress of the material.

- If $\eta_l > 2\ \eta_c$, then the circumferential joint becomes critical and hence the following expression governs the design

$$\sigma_l\,\eta_c = \frac{p\,d}{4t} \qquad (20)$$

where σ is equated to the safe or permissible stress of the material.

Example 1

What pressure may be allowed in a cylindrical boiler 2.5 m internal diameter with plates 20 mm thick, if the safe intensity of tensile stress is 65 MPa.

Given : $d = 2500mm$ $t = 20mm$

Since $\sigma_C > \sigma_l$, the safe intensity of stress should be equated to σ_C

Hence $\sigma_C = \sigma_{safe} = 65$ MPa

We have $\sigma_C = \dfrac{pd}{2t}$

Hence, $p \leq \dfrac{\sigma_c 2\,t}{d} \leq 1.04MPa$

Thus the safe allowable internal pressure in the cylinder is 1.04 MPa.

Example 2

Determine the minimum thickness of the plate required for boilers of internal diameter 1.5 m and internal pressure of 1 MPa if the efficiency of riveted joints is 60 %. The permissible stress in steel plate is 150 MPa.

Given : $\eta_l = \eta_C = 0.6$.

This satisfies the condition $\eta_l < 2\,\eta_C$

Hence the following expression (eq. 19) governs the design for the given data

$\sigma_c \eta_l = \dfrac{p\,d}{2t}$ where $\sigma_C = \sigma_{safe} = 150$ MPa

Hence, $t \geq \dfrac{p\,d}{2\sigma_c} x \dfrac{1}{\eta_l} \geq 8.33mm$

Thus the minimum thickness of the plate is 8.33 mm

Example 3

A thin cylinder of internal diameter 1m and thickness 15 mm is made of number of sheets which are riveted together. If the efficiency the longitudinal joint is 90% and that of the circumference joint is 40%, determine the safe allowable internal pressure. Assume the allowable tensile stress as 50 MPa.

Given : $\eta_l = 0.9$ *and* $\eta_C = 0.4$.

This satisfies the condition $\eta_l < 2\eta_C$

Hence the following expression (eq. 20) governs the design for the given data

$$\sigma_c \eta_c = \frac{p\,d}{4t} \qquad \text{where } \sigma_l = \sigma_{allowable} = 150 \text{ MPa}$$

Hence, $$p \leq \frac{f_l\, 4\, t\, \eta_C}{d} \leq 1.2\,MPa$$

Thus the safe allowable internal pressure = 1.2 MPa.

Example 4

A thin cylindrical shell 1m in diameter and 3m long has a metal thickness of 10 mm. It is subjected to an internal fluid pressure of 3 MPa. Determine the changes in length, diameter and volume. Also find the maximum shear stress in the shell. Assume E_S = 210 GPa and υ = 0.3.

Given: d = 1000mm, l = 3000 mm, t = 10 mm, p = 3MPa, E = 210 GPa and υ = 0.3

a) <u>Change in length</u>

The longitudinal strain is given by (eq. 13)

$$\varepsilon_l = \frac{p\,d}{2\,t\,E}\left(\frac{1}{2} - \nu\right)$$

Substituting the data $\varepsilon_l = 1.43\,x10^{-4}$

Since $$\varepsilon_l = \frac{\delta\, l}{l}$$

Change in length, $\delta\, l = \varepsilon_l\, l = 0.43 mm$

b) <u>Change in diameter</u>

The circumferential strain (or diametral strain) is given by (eq. 10)

$$\varepsilon_C = \frac{p\,d}{2\,t\,E}\left(1 - \frac{\nu}{2}\right)$$

Substituting the data $\varepsilon_C = 6.1\,x10^{-4}$

Since $$\varepsilon_c = \frac{\delta\, d}{d}$$

Change in diameter, $\delta\, d = \varepsilon_C\, d = 0.61 mm$

c) <u>Change in volume</u>

The internal volume V of the cylinder is given by

$$v = \left(\frac{\Pi}{4}\, x\, d^2\right)(l) = 2.356\, x10^9\, mm^3$$

The volumetric strain is given by (eq. 16)

$$\varepsilon_v = 2\,\varepsilon_C + \varepsilon_l$$

Substituting $\varepsilon_C = 6.1\,x10^{-4}$ and $\varepsilon_l = 1.43x10^{-4}$

Volumetric strain, $\varepsilon_v = 13.63\,x10^{-4}$

Since $\varepsilon_v = \dfrac{\delta v}{v}$

Change in volume, $\delta v = \varepsilon_v V = 3211493.09\text{mm}^3$

d) <u>Max Shear Stress</u>

The maximum shear stress is given by (eq. 8)

$$\sigma_{s\,\max} = \frac{p\,d}{8\,t} = 37.5\,MPa$$

3.4 THICK CYLINDER THEORY

In thin cylinders, the average circumferential stress (or hoop stress) is nearly equal to the maximum circumferential stress and hence the distribution of this stress over the cylinder wall is considered to be uniform. But in thick cylinders, the distribution of circumferential stress is considered to be non-uniform, as the average circumferential stress is much smaller than the maximum circumferential stress. Moreover, the variation of circumferential stress in thick cylinder is observed to be non-linear. Further, the radial stress which is neglected in thin cylinders is accounted in thick cylinders since its magnitude is considerable.

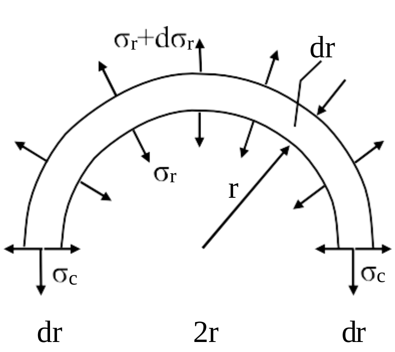
3.4.1 Assumptions

The problem of determining the circumferential stress σ_c and radial stress σ_r at any point on a thick walled cylinder in terms of the applied pressures and dimensions was first solved by the French elastician, Gabriel Lame in 1833. The following assumptions were made during the analysis.

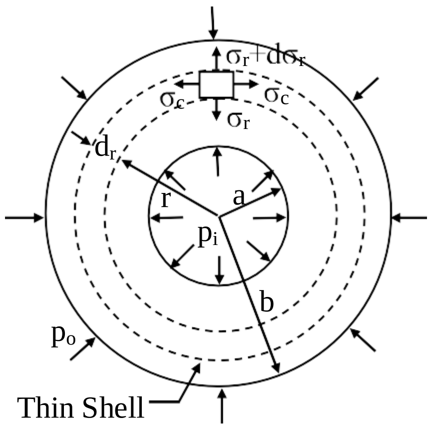
1. The material is homogeneous, isotropic and elastic.
2. The stresses are within the proportionality limit.
3. The longitudinal strain remains constant for all fibres.
4. The circumferential stress (or hoop stress) is considered to vary across the wall thickness. It is maximum at the inner surface and minimum at the outer surface.

3.4.2 Expressions for Circumferential and Radial Stresses in Thick Cylinders [Lame's Equations]

Consider a thick cylinder of internal radius 'a' and external radius 'b' subjected to a uniformly distributed internal pressure 'p_i' and external pressure 'p_o' as shown in Fig. 7. The thick cylinder is assumed to be composed of a number of thin shells as shown.



Free body diagram



Consider the free body diagram of the half-section of a typical thin shell, the radius of which is 'r' and thickness 'dr', as shown in Fig 7. The circumferential stress in this shell is σ_c. The radial stress on the inner surface is σ_r and that on the outer surface is '$\sigma_r + d\sigma_r$', where '$d\sigma_r$' is the increment in 'σ_r' due to the variation of pressure across the cylinder body. The radial stresses are assumed (incorrectly) to be tensile, so a negative result for 'σ_r' will denote compression. Let 'σ_r' be the

Considering the free body diagram of the half section, and applying the equilibrium equation

$$\Sigma V = 0 \; [\uparrow + ve]$$

$$(\sigma_r + d\sigma_r)\,[2\,(r + dr)] - \sigma_r\,(2r) - 2\,\sigma_c(dr) = 0$$

Ignoring very small terms, the above equation reduces to

$$\sigma_r \,.\, r + \sigma_r \,.\, dr + r \,.\, d\,\sigma_r - \sigma_r \,.\, r - \sigma_c \,.\, dr = 0$$

On rearranging,

$$r \,.\, \frac{d\sigma_r}{dr} + \sigma_r - \sigma_c = 0 \qquad (21)$$

The element in the wall of a thick shell will be subjected to all the three stresses, namely, circumferential stress 'σ_c', longitudinal stress 'σ_l' and adial stress 'σ_r'. Using Hooke's law for triaxial state of stress, the longitudinal strain ε_l is given by

$$\varepsilon_l = \frac{\sigma_l}{E} - \upsilon\left(\frac{\sigma_c}{E}\right) - \upsilon\left(\frac{\sigma_r}{E}\right)$$

or $$\varepsilon_l = \frac{1}{E}\left[\sigma_l - \upsilon\,(\sigma_c + \sigma_r)\right]$$

In case of thick cylinders, longitudinal strain 'ε_l' is a constant and hence 'σ_l' is a constant. Further, E and υ are also constants. Hence it implies that $(\sigma_c + \sigma_r)$ should also be a constant.

Let $$\sigma_c + \sigma_r = 2A \quad \text{where A is a constant} \tag{22}$$

Adding eqs. (21) and (22)

$$r \cdot \frac{d\sigma_r}{dr} + 2\sigma_r = 2A$$

or $$r \cdot \frac{d\sigma_r}{dr} = 2\,(A - \sigma_r)$$

Separating the variables

$$\frac{d\sigma_r}{(A-\sigma_r)} = 2 \cdot \frac{dr}{r}$$

On integrating

$$-\log_e (A - \sigma_r) = 2\log_e . r + C \qquad \text{where C is a constant}$$

$$\log_e [(A - \sigma_r) . r^2] = -C$$

$$\log_e [(A - \sigma_r) . r^2] = \log_e B \tag{23}$$

where $\log_e B = -C$, and B is another constant.

From eq. (23) $(A - \sigma_r)\, r^2 = B$

$$\therefore \sigma_r = A - \frac{B}{r^2} \tag{24}$$

Substituting eq. (24) in eq. (22)

$$\sigma_c + \left[A - \frac{B}{r^2}\right] = 2A$$

$$\therefore \sigma_c = A + \frac{B}{r^2} \tag{25}$$

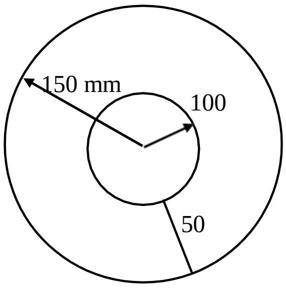
Note 1: In equations (24) and (25) 'A' and 'B' are constants which depend on the boundary conditions. Hence the values of 'A' and 'B' are different for different examples. Further, 'r' is the radial distance to a point in the wall of the cylinder at which the stresses 'σ_c' and 'σ_r' are to be determined.

Note 2: From equations (24) and (25), it can be observed that σ_c is greater than σ_r. Further 'σ_c' is maximum when 'r' is minimum (ie, at internal surface). Hence, if maximum allowable stress of the material is given it should be equated to circumferential stress at the internal surface.

Note 3: From equations (24) and (25), it can be seen that both 'σ_c' and 'σ_r' depend on r^2. Hence the variation of these stresses is non-linear.

Example 5

A thick cylindrical pipe of external diameter 300 mm and thickness 50 mm is subjected to an internal fluid pressure of 40 MPa and an external pressure of 2.5 MPa. Calculate the maximum and minimum intensities of circumferential and radial stresses in the pipe section. Sketch the variation of stresses across the pipe section.



Given: Thickness t = 50 mm

External diameter = 300 mm.

Hence, external radius b = 150 mm

Internal radius a = b – t = 100 mm

The Lame's expressions for thick cylinder are

$$\sigma_c = A + \frac{B}{r^2} \qquad (26)$$

and $$\sigma_r = A - \frac{B}{r^2} \qquad (27)$$

The constants 'A' and 'B' are evaluated using the known boundary conditions.

Boundary condition 1:

The cylinder is subjected to an internal pressure of 40 MPa.

Hence @ r = 100 mm, 'σ_r' = – 40 MPa (Compressive)

From (26) $$-40 = A - \frac{B}{(100)^2} \qquad (28)$$

Boundary condition 2:

The cylinder is subjected to an external pressure of 2.5 MPa.

Hence @ r = 150 mm, 'σ_r' = – 2.5 MPa (Compressive)

From (26) $$-2.5 = A - \frac{B}{(150)^2} \tag{29}$$

Solving eqs. (28) and (29); A = 27.5 and B = 675000

Hence eqs. (26) and (27) take the form

$$\sigma_c = 27.5 + \frac{675000}{r^2} \tag{30}$$

$$\sigma_r = 27.5 - \frac{675000}{r^2} \tag{31}$$

From eq. (30) the distribution of hoop stress can be determined. Hence

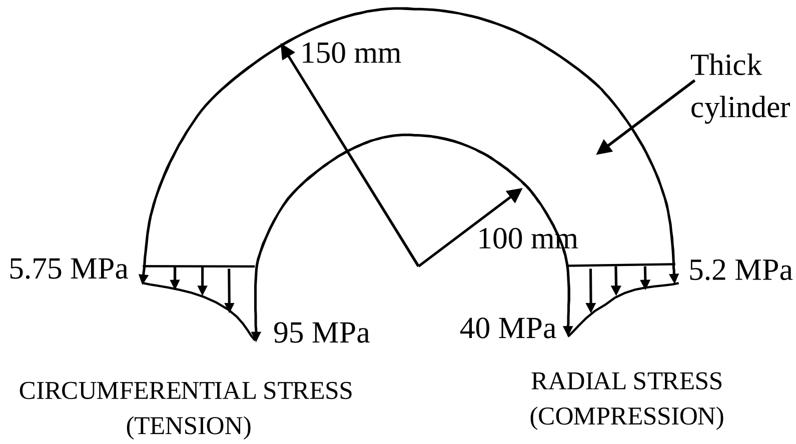
@ r = 100 mm, σ_c = 95 MPa (Tensile)

@ r = 150 mm, σ_c = 57.5 MPa (Tensile)

From eq. (31) the distribution of radial stress can be determined. Hence

@ r = 100 mm, σ_r = – 40 MPa (given) (Compressive)

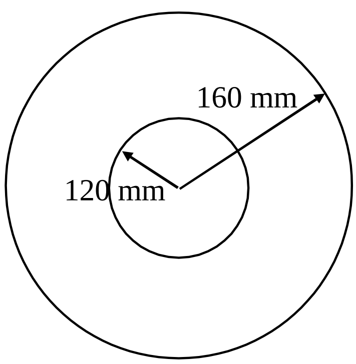
@ r = 150 mm, σ_r = – 2.5 MPa (given) (Compressive)



Variation of stresses across wall thickness

Example 6

A thick cylindrical pipe of internal radius 120 mm and external radius 160 mm is subjected to an internal fluid pressure of 12 MPa. Determine the hoop stress in the cross section. What is the percentage



error if the maximum hoop stress is found from the equation of thin pipes?

Given: Internal radius a = 120 mm, and External radius b = 160 mm

The Lame's expressions for thick cylinder are

$$\sigma_c = A + \frac{B}{r^2} \qquad (32)$$

and $$\sigma_r = A - \frac{B}{r^2} \qquad (33)$$

The constants 'A' and 'B' are evaluated using the known boundary conditions.

Boundary condition 1:

The cylinder is subjected to an internal pressure of 12 MPa.

Hence @ r = 120 mm, σ_r = – 12 MPa (Compressive)

From (33) $$-12 = A - \frac{B}{(120)^2} \qquad (34)$$

Boundary condition 2:

The cylinder is not subjected to any external pressure.

Hence @ r = 160 mm, σ_r = 0

From (33) $$0 = A - \frac{B}{(160)^2} \qquad (35)$$

Solving eqs. (34) and (35); A = 15.43 and B = 394971.43

Hence eqs. (32) and (33) take the form

$$\sigma_c = 15.43 + \frac{394971.43}{r^2} \qquad (36)$$

$$\sigma_r = 15.43 - \frac{394971.43}{r^2} \qquad (37)$$

From eq. (36) the distribution of hoop stress can be determined. Hence

@ r = 120 mm, σ_c = 42.86 MPa (Tensile)

@ r = 160 mm, σ_c = 30.86 MPa (Tensile)

From eq. (37) the distribution of radial stress can be determined. Hence

@ r = 120 mm, σ_r = 12 MPa (given) (Compressive)

@ r = 160 mm, σ_r = 0 (given) (Compressive)

The maximum circumferential stress obtained (at inner surface) using Lame's equations is

$\sigma_c = 42.86$ MPa

Using thin cylinder theory, the circumferential stress is obtained as

$$\sigma_c = \frac{p\,d}{2\,t} = \frac{(12)(240)}{2\,(40)} = 36.0\ MPa$$

Therefore, percentage error = $\dfrac{42.86 - 36.00}{42.86} x100 = 16\ \%$

Example 7

A thick cylinder of internal diameter 200 mm is subjected to an internal fluid pressure of 40 MPa. If the allowable stress in tension for the material is 120 MPa find the thickness of the cylinder.

Given: Internal diameter = 200 mm. Hence, internal radius a = 100 mm

The Lame's expressions for thick cylinder are

$$\sigma_c = A + \frac{B}{r^2} \qquad (38)$$

and $$\sigma_r = A - \frac{B}{r^2} \qquad (39)$$

The constants 'A' and 'B' are evaluated using the known boundary conditions.

<u>Boundary condition 1</u>:

The cylinder is subjected to an internal pressure of 40 MPa.

Hence @ r = 100 mm, σ_r = – 40 MPa (Compressive)

From (39) $$-40 = A - \frac{B}{(100)^2} \qquad (40)$$

<u>Boundary condition 2</u>:

It is known that σ_c is always more than σ_r. Further σ_c is maximum at inner surface. Hence equate the given allowable stress to σ_c at inner surface.

Hence @ r = 100 mm, σ_c = 120 MPa (tensile)

From (38) $$120 = A + \frac{B}{(100)^2} \qquad (41)$$

Solving eqs. (40) and (41); A = 40 and B = 800000

To find thickness, apply the third boundary condition.

Boundary condition 3:

The cylinder is subjected to zero external pressure.

Hence @ r = (100 + t) mm, $\sigma_r = 0$

From (39) $$0 = 40 - \frac{800000}{(100+t)^2} \qquad (42)$$

From eq. (42), thickness of cylinder is

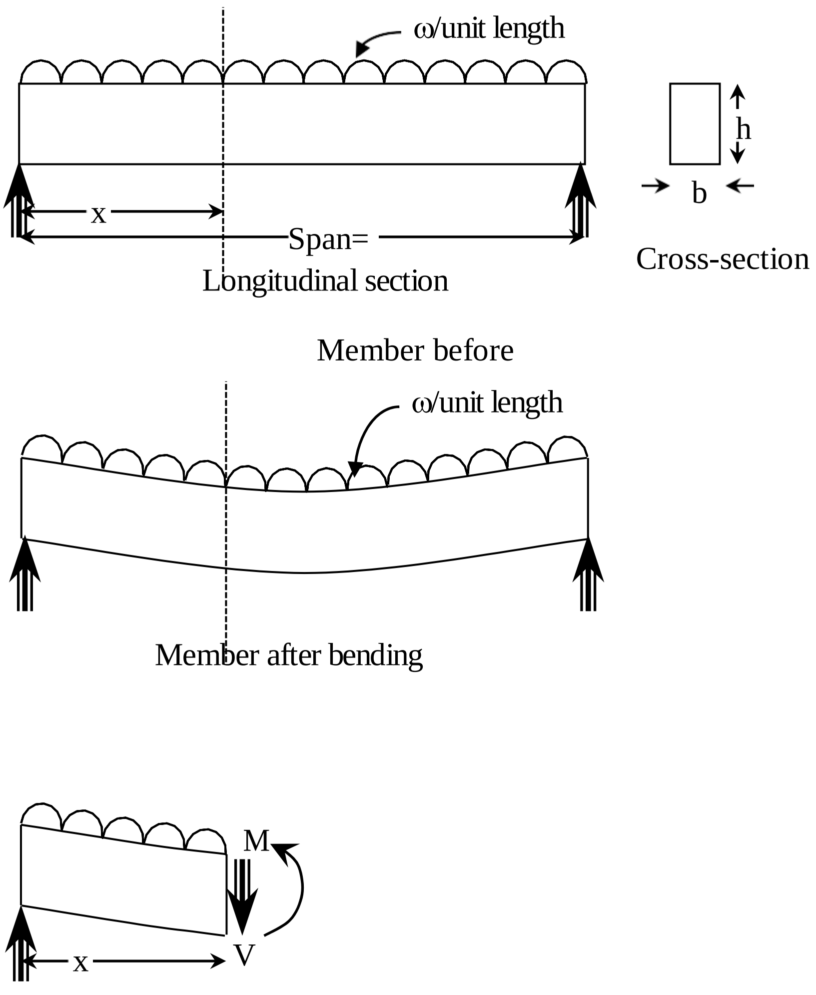
t = 141.42 mm

BENDING MOMENT AND SHEAR FORCES

INTRODUCTIO N

Beam is a structural member, which has negligible cross- section compared to its length. It carries load perpendicular to the axis in the plane of the beam. Due to the loading on the beam, the beam deforms and is called as deflection in the direction of loading. This deflection is due to bending moment and shear force generated as resistance to the bending. Bending Moment is defined as the internal resistance moment to counteract the external moment due to the loads and mathematically it is equal to algebraic sum of moments of the loads acting on one side of the section. It can also be defined as the unbalanced moment on the beam at that section.

Shear force is the internal resistance developed to counteract the shearing action due to external load and mathematically it is equal to algebraic sum of vertical loads on one side of the section and this act tangential to cross section. These two are shown in Fig 3.01 (a).



Unbalanced Moment = Bending Moment (M) & Unbalanced Force = Shear Force

Fig. 3.01 (a)

For shear force Left side Upward force to the section is Positive (LUP) and Right side Upward force to the section is Negative (RUN) as shown in Fig. 3.01 (b).

For Bending Moment, Moment producing sagging action to the beam or clockwise moment to the left of the section and anti- clockwise moment to the right of the section is treated as positive and Moment producing hogging action to the beam or anti- clockwise moment to the left of the section and clockwise moment to the right of the section is treated as Negative as shown in Fig. 3.01(b).

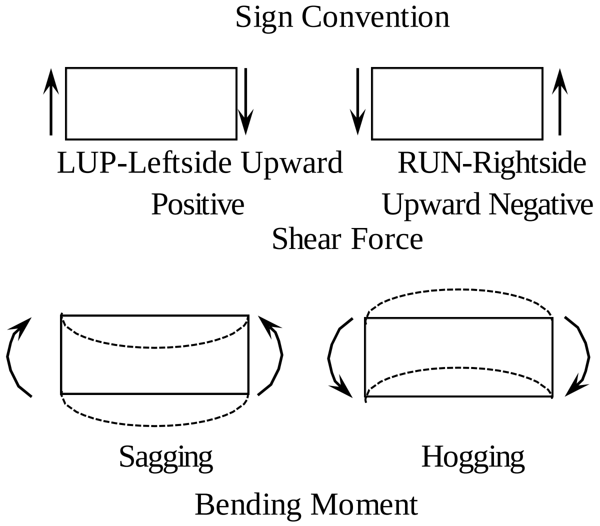


Fig. 3.01 (b)

Elastic Curve

Generally the beam is represented by a line and the beam bends after the loading. The depiction of the bent portion of the beam is known as elastic curve.

The shape of the elastic curve is the best way to find the sign of the Bending Moment as shown in the Fig. 3.02

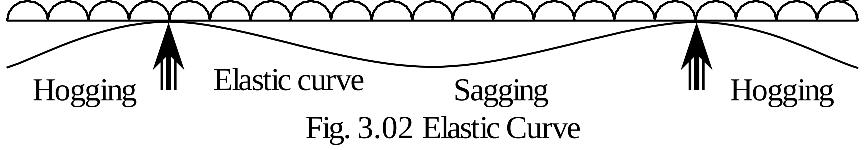


Fig. 3.02 Elastic Curve

Support Reactions:

The various structural members are connected to the surroundings by various types of supports .The structural members exert forces on supports known as action. Similarly supports exert forces on structural members known as reaction.

A beam is a horizontal member, which is generally placed on supports.

The beam is subjected to the vertical forces known as action. Supports exe rt forces on beam known as reaction.

Types of supports:

1) Simple supports
2) Roller supports
3) Hinged or pinned supports
4) Fixed supports

1) Simple supports:

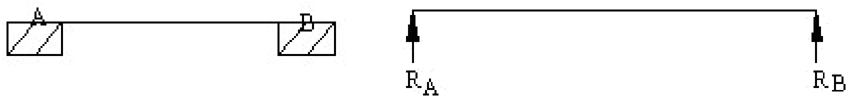


Fig. 3.03

Simple supports are those supports, which exert reactions perpendicular to the plane of support. It restricts the translation of body in one direction only, but not rotation.

2) Roller supports:

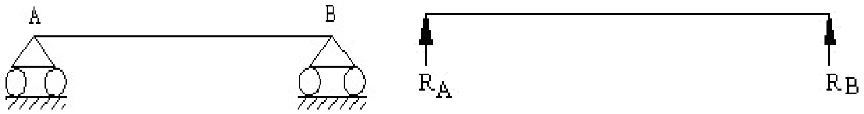


Fig. 3.04

Roller supports are the supports consisting of rollers which exert reactions perpendicular to the plane of the support. They restrict translation along one direction and no rotation.

3) Hinged or Pinned supports:

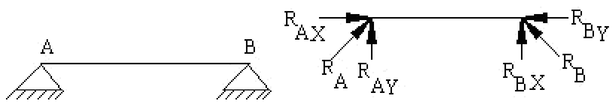


Fig. 3.05

Hinged supports are the supports which exert reactions in any direction but for our convenient point of view it is resolved in to two components. Therefore hinged supports restrict translation in both directions. But rotation is possible.

4) Fixed supports:

Fixed supports are those supports which restricts both translation and rotation of the body. Fixed supports develop an internal moment known as restraint moment to prevent the rotation of the body.

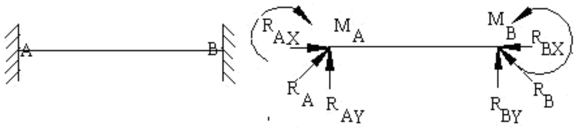


Fig. 3.06

Types of Beams:-

1) Simply supported Beam:

Fig. 3.07

It is a beam which consists of simple supports. Such a beam can resist forces normal to the axis of the beam.

2) **Continuous Beam**:

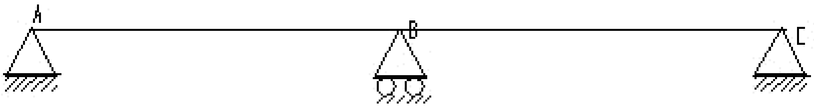


Fig. 3.08

It is a beam which consists of three or more supports.

3) **Cantilever beam**:

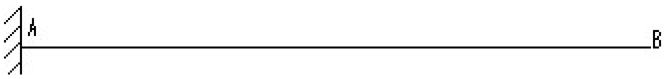


Fig. 3.09

It is a beam whose one end is fixed and the other end is free.

3) Propped cantilever Beam:

It is a beam whose one end is fixed and other end is simply supported.

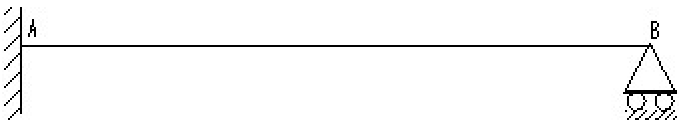


Fig. 3.10

4) Overhanging Beam:

It is a beam whose one end is exceeded beyond the support.

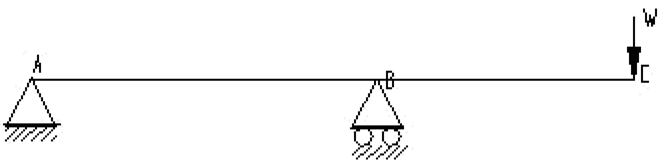


Fig.3.11

Types of loads:

1) **Concentrated load**: A load which is concentrated at a point in a beam is known as concentrated load.

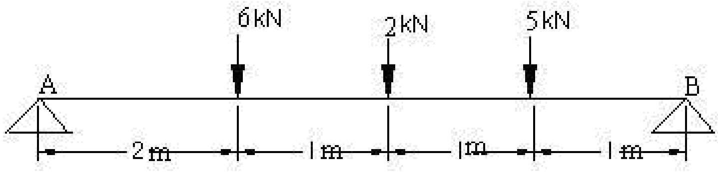


Fig. 3.12

2) **Uniformly Distributed load**: A load which is distributed uniformly along the entire length of the beam is known as Uniformly Distributed Load.

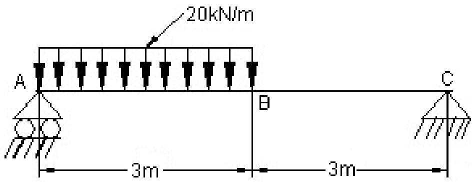


Fig. 3.13

Convert the U.D.L. into point load which is acting at the centre of particular span

Magnitude of point load=20KN/mx3m=60kN

3) **Uniformly Varying load**: A load which varies with the length of the beam is known as Uniformly Varying load

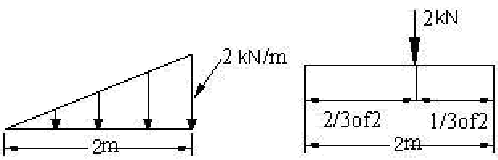


Fig. 3.14

Magnitude of point load=Area of triangle and which is acting at the C.G. of triangle.

Problems on Equilibrium of coplanar non concurrent force system. Tips to find the support reactions:

1) In coplanar concurrent force system, three conditions of equilibrium can be applied namely

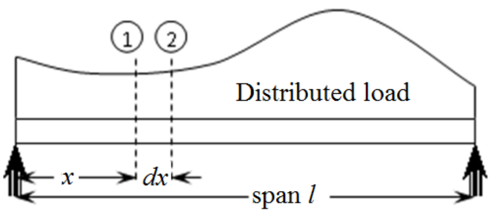
$\sum Fx = 0$, $\sum Fy = 0$ and $\Sigma M = 0$

2) Draw the free body diagram of the given beam by showing all the forces and reactions acting on the beam

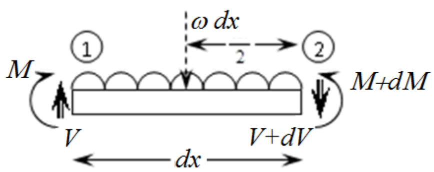
3) Apply the three conditions of equilibrium to calculate the unknown reactions at the supports. **<u>Determinate</u> <u>structures</u>** are those which can be solved with the fundamental equations of equilibrium. i.e. the 3 unknown reactions can be solved with the three equations of equilibrium.

Relationship between Uniformly distributed load (udl), Shear force and Bending Moment.

Consider a simply supported beam subjected to distributed load ω which is a function of x as shown in Fig. 3.15(a). Consider section 11 at a distance x from left support and another section 22 at a small distance dx from section 11. The free body diagram of the element is as shown in Fig. 3.15(b). To the left of the section 11 the internal force V and the moment M acts in the +ve direction. To the right of the section 22 the internal force and the moment are assumed to increase by a small amount and are respectively $V+dV$ and $M+dM$ acting in the +ve direction.



Longitudinal section of the loaded beam



Free body diagram of the element of the beam

For the equilibrium of the system, the algebraic sum of all the vertical forces must be zero.

$\rightarrow +\text{ve}\sum V = 0;$

$$V - \omega dx - (V + dV) = 0$$

$$-\omega dx - dV = 0$$

$$-\omega = \frac{dV}{dx} \qquad \ldots(01)$$

Eq. 01 the udl at any section is given by the negative slope of shear force with respect to distance x or negative udl is given by the rate of change of shear force with respect to distance x.

Within a limit of distributed force ω_1 and ω_2 over a distance of a, shear force is written as

$$V = \int_{\omega_1}^{\omega_2} -\omega dx$$

For the equilibrium of the system, the algebraic sum Moments of all the forces must be zero. Taking moment about section 22

$\sum M = 0;$

$$M + Vdx - (\omega dx)\left(\frac{dx}{2}\right) - (M + dM) = 0$$

Ignoring the higher order derivatives, we get

$$Vdx - dM = 0$$

$$\text{or } V = \frac{dM}{dx} \qquad 02$$

Eq. 02 shows the shear force at any section is given by rate of change in bending moment with respect to distance x.

Within a limit of distributed force ω_1 and ω_2 and shear force V_1 and V_2 over a distance of a, we can write bending moment as

$$M = \int_{V_1}^{V_2} Vdx$$

Point of contra flexure or point of inflection.

These are the points where the sign of the bending moment changes, either from positive to negative or from negative to positive. The bending moment at these points will be zero.

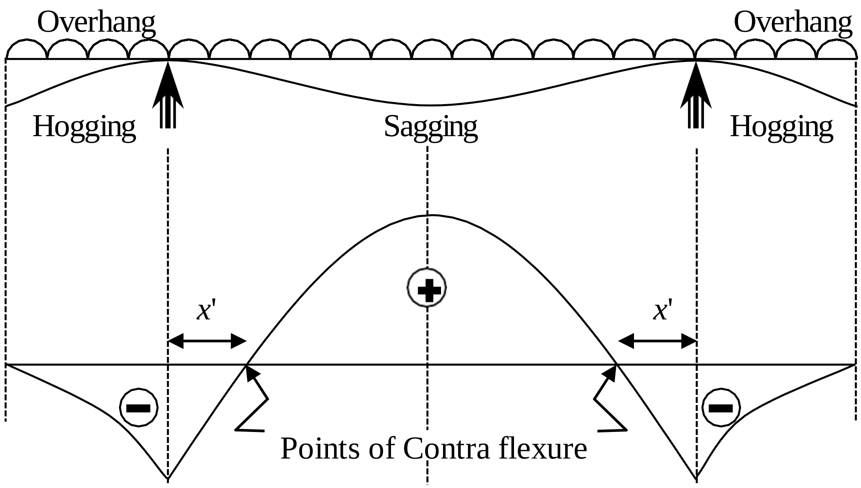


Fig. 3.16 Bending Moment Diagram

Procedure to draw Shear Force and Bending Moment Diagram

- Determine the reactions including reactive moments if any using the conditions of equilibrium viz. $\sum H = 0$; $\sum V = 0$; $\sum M = 0$

Shear Force Diagram (SFD)

- Draw a horizontal line to represent the beam equal to the length of the beam to some scale as zero shear line.
- The shear line is vertical under vertical load, inclined under the portion of uniformly distributed load and parabolic under the portion of uniformly varying load. The shear line will be horizontal under no load portion. Remember that the shear force diagram is only concerned with vertical loads only and not with horizontal force or moments.
- Start from the left extreme edge of the horizontal line (For a cantilever from the fixed end), draw the shear line as per the above described method. Continue until all the loads are completed and the check is that the shear line should terminate at the horizontal line.
- The portion above the horizontal line is positive shear force and below the line is negative shear force.
- To join the shear line under the portion of uniformly varying load, which is a parabola, it is to be remembered that the parabola should be tangential to the horizontal if the

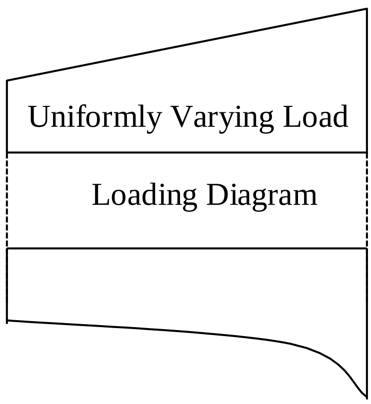


Fig. 3.17 Shear Force Diagram

corresponding load at the loading diagram is lesser and will be tangential to vertical if the corresponding load at the loading diagram is greater.

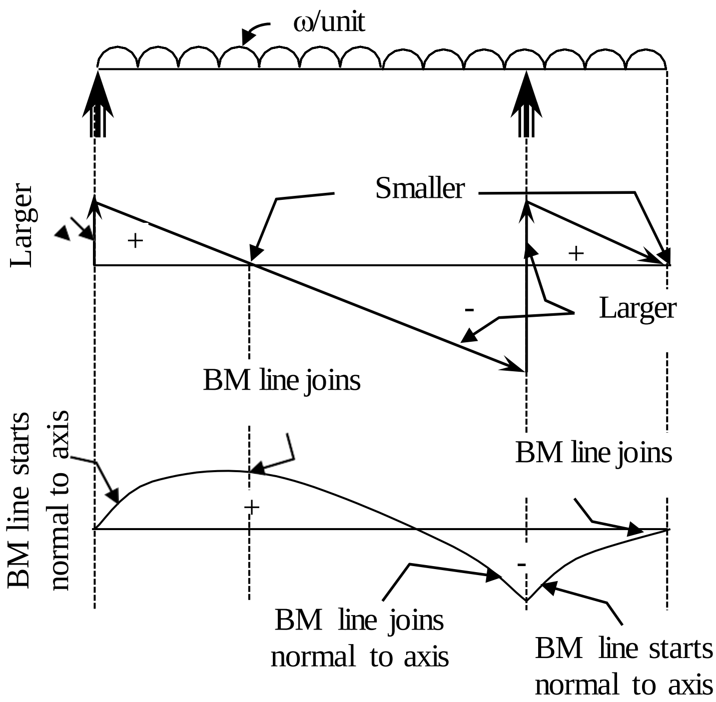


Fig. 3.18 SFD, BMD and Loading Diagrams

Bending Moment Diagram (BMD)

- Draw a horizontal line to represent the beam equal to the length of zero shear line under the SFD.
- The Bending Moment line is vertical under the applied moment, inclined or horizontal under the no load portion, parabolic under the portion of uniformly distributed load and cubic parabola under the portion of uniformly varying load.
- Compute the Bending Moment values as per the procedure at the salient points.
- Bending Moment should be computed just to the left and just to the right under section where applied moment is acting. i.e. M_{AL} and M_{AR}. Once the applied moment is to be ignored and next the moment is to be considered as per the sign convention.
- Draw these values as vertical ordinates above or below the horizontal line corresponding to positive or negative values.

- Start the Bending Moment line from the left extreme edge of the horizontal line, draw as per the above described method under prescribed loading conditions. Continue until the end of the beam and the check is that the line should terminate at the horizontal line.
- The portion above the horizontal line is positive Bending Moment and below the line is negative Bending Moment.
- Locate the point of Maximum Bending Moment. It occurs at the section where Shear Force is zero.
- Locate the Point of Contra flexure where the Bending Moment line crosses the horizontal line. i.e. the sign of Bending Moment line changes its sign.

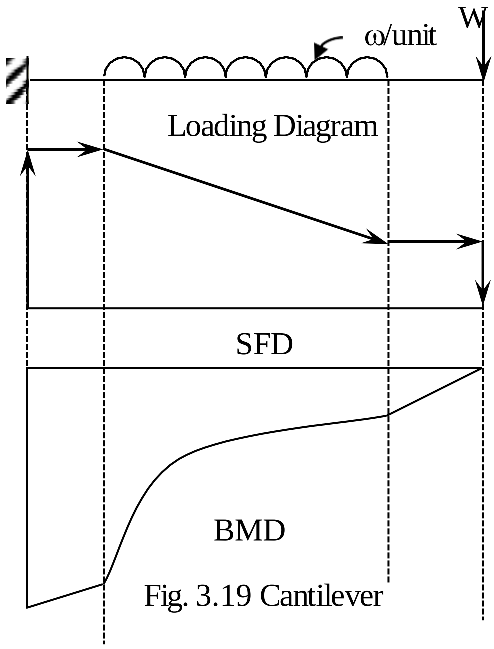


Fig. 3.19 Cantilever

To join the Bending Moment line under the portion of uniformly distributed load which is a parabola, it is to be remembered that the parabola should be tangential to the horizontal if the corresponding shear force value at the loading diagram is lesser and will be tangential to vertical if the corresponding shear force line at the shear force diagram is greater as shown in Fig. 3.17.

In case of the beam being a ***Cantilever***, start the Shear force from the fixed end. i.e. arrange the cantilever such that the fixed end is towards left end.

Problems

STANDARD PROBLEMS

Eccentric Concentrated Load

Consider a simply supported beam of span *l* with an eccentric point load W acting at a distance *a* from support as shown in Fig. 3.20

The reactions can be obtained from the equations of equilibrium

(Write the Upward acting forces on one side and downward acting forces on the other side of the equation to avoid confusion among sign convention).

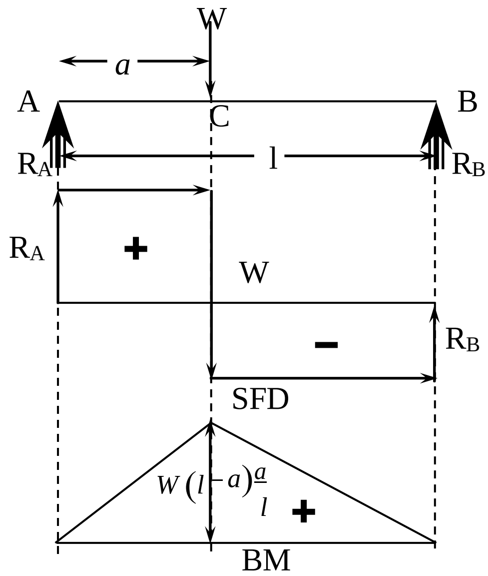


Fig. 3.20 SS with Point load

$\sum V_A = 0$; $R_A + R_B = W$ (01)

Taking moments about A,

$\sum M_A = 0$;

(Write the clockwise moments on one side and anti-clockwise moments on the other side of the equation to avoid confusion among sign convention).

$(R_B)(l) = (W)(a)$

$$R_B = \frac{Wa}{l}$$

Similarly Taking moments about B,

$\sum M_B = 0$;

$(R_A)(l) = (W)(l—a)$

$$R_A = \frac{W(l-a)}{l}$$

Check

To check the computations, substitute in Eq. 01, we have

$R_A + R_B = \frac{Wa}{l} + \frac{W(l-a)}{l} = W\left[\frac{a+l-a}{l}\right] = W$ and hence OK.

Shear Force Values

$$V_A = 0 + R_A = \frac{W(l-a)}{l}$$

$$V_C = \frac{W(l-a)}{l}$$

$$V_C = \frac{W(l-a)}{l} - W = -\frac{Wa}{l}$$

$$V_B = -\frac{Wa}{l}$$

$$V_B = -\frac{Wa}{l} + \frac{Wa}{l} = 0$$

Bending Moment Values

Note: The Bending Moment will always will be zero at the end of the beam unless there is an applied moment at the end.

$M_A = 0$

$M_B = 0$

$$M_C = (R_A)a = \frac{W(l-a)}{l} \times a = W(l-a)\frac{a}{l} \text{ also}$$

$$M_C = (RB)(l-a) = \left(\frac{Wa}{l}\right) \times (l-a) = W(l-a)\frac{a}{l}$$

Uniformly Distributed Load

Consider a simply supported beam of span l with an uniformly distributed load ω/m acting over the entire span as shown in Fig. 3.35

The reactions can be obtained from the conditions of equilibrium.

As the loading is symmetrical

$R_A = R_B$ and hence

$\sum V_A = 0$; $R_A + R_B = 2\ R_A = 2\ R_B = \omega \times l$

$$R_A = R_B = \frac{\omega l}{2}$$

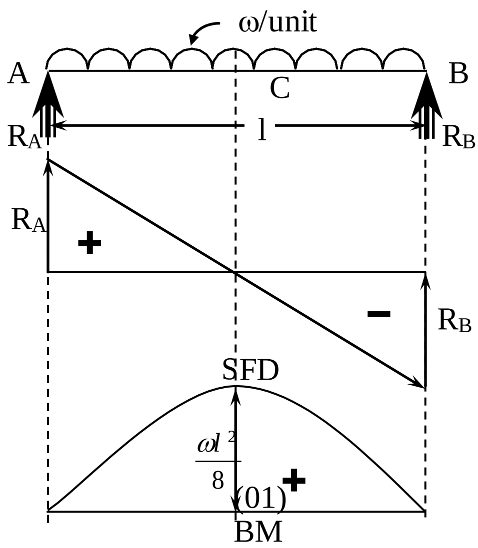


Fig. 3.21 SS with UDL

Shear Force Values

$$V_A = R_A = \frac{\omega l}{2}$$

$$V_B = \frac{\omega l}{2} - \omega l = -\frac{\omega}{2}$$

Shear Force at Midsection will be

$$V_C = \frac{\omega l}{2} - \frac{\omega l}{2} = 0$$

Bending Moment Values

$M_A = 0$

$M_B = 0$

$$M_C = (R_A)\frac{l}{2} = \left[\frac{\omega l}{2}\right]\left[\frac{l}{2}\right] = \frac{\omega l^2}{4}$$

Uniformly Varying Load

Consider a simply supported beam of span l with an uniformly varying load ω/m acting over the entire span as shown in Fig. 3.24

The reactions can be obtained from the conditions of equilibrium.

$\sum V_A = 0;$

$$R_A + R_B = \left(\frac{\omega l}{2}\right) \qquad (01)$$

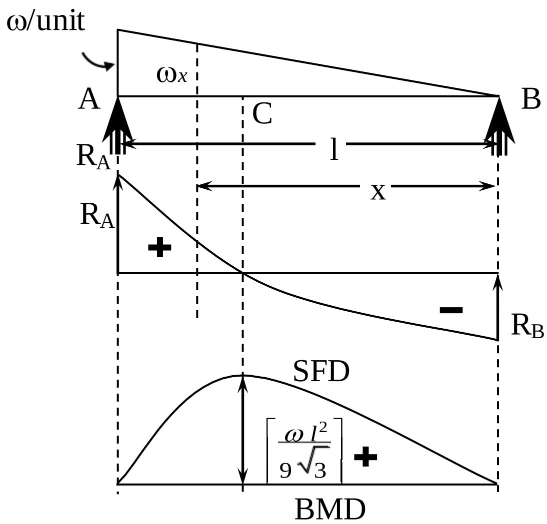


Fig. 3.22 SS with UVL

Taking moments about A,

$\sum M_A = 0;$

$$R_B \times l = \left(\frac{\omega l}{2}\right)\left(\frac{l}{3}\right) = \frac{\omega l^2}{6}$$

$$R_B = \frac{\omega l}{6}$$

Taking moments about B,

$\sum M_B = 0;$

$$R_A \times l = \left(\frac{\omega l}{2}\right)\left(\frac{2l}{3}\right) = \frac{\omega l^2}{3}$$

$$R_A = \frac{\omega l}{3}$$

Check

To check the computations, substitute in Eq. 01, we have

$$R_A + R_B = \left(\frac{\omega l}{6}\right) + \left(\frac{\omega l}{3}\right) = \frac{\omega l}{2}$$

Hence O.K.

Shear Force Values

$$V_A = R_A = \frac{\omega l}{3}$$

$$V_B = \frac{\omega l}{3} - \frac{\omega l}{2} = -\frac{\omega l}{6} \text{ and } V_B = -\frac{\omega l}{6} + \frac{\omega l}{6} = 0$$

Location of Zero Shear Force

Consider a section at a distance x from left support and load intensity at that section ω_x is given by $\omega_x = \left(\frac{x}{l}\right)\omega$

and Shear Force at that section is given by

$$V_x = \frac{1}{2}\omega_x \times x - R_B = \left(\frac{\omega x^2}{2l}\right) - \left(\frac{\omega l}{6}\right) \Rightarrow 0 \text{ or } x = \frac{l}{\sqrt{3}}$$

Bending Moment Values

$M_A = 0$

$M_B = 0$

Bending Moment will be maximum at Zero Shear Force and

$$M_c = (R_B)\, x - \left[\frac{1}{2} \times \omega_x \times x\right]\left(\frac{x}{3}\right) = \left[\frac{\omega l}{6}\right] \times x - \left[\frac{\omega x^3}{6l}\right]$$

$$= \left[\frac{\omega l}{6}\right] \times \left[\frac{l}{\sqrt{3}}\right] - \left[\frac{\omega}{6l}\right]\left[\frac{l}{\sqrt{3}}\right]^3$$

$$= \left[\frac{\omega l^2}{6\sqrt{3}}\right]\left(1 - \frac{1}{3}\right) = \left[\frac{\omega l^2}{9\sqrt{3}}\right]$$

Cantilever with Point Load

The reactions can be obtained from the conditions of equilibrium.

$\sum V_A = 0;\ R_A = W$

Taking moments about A,

$M_A = -W(l - a)$

Shear Force Values

$V_B = 0$

$V_C = 0$

$V_C = 0 - W = -W$

$V_A = -W$

$V_A = -W + W = 0$

Bending Moment Values

$M_B = 0$

$M_C = 0$

$M_A = -W(l - a)$

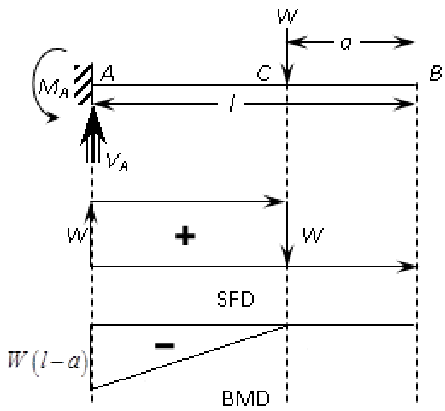


Fig. 3.33 Cantilever with Point Load

Cantilever with Uniformly Distributed Load (UDL)

The reactions can be obtained from the conditions of equilibrium.

$\sum V_A = 0;\ R_A = \omega l$

Taking moments about A,

$$M_A = -\omega l \times \left(\frac{l}{2}\right) = -\frac{\omega l^2}{2}$$

Shear Force Values

$V_B = 0$

$V_A = -\omega l$

$V_A = -\omega l + \omega l = 0$

Bending Moment Values

$M_B = 0$

$$M_A = -\omega l \times \left(\frac{l}{2}\right) = -\frac{\omega l^2}{2}$$

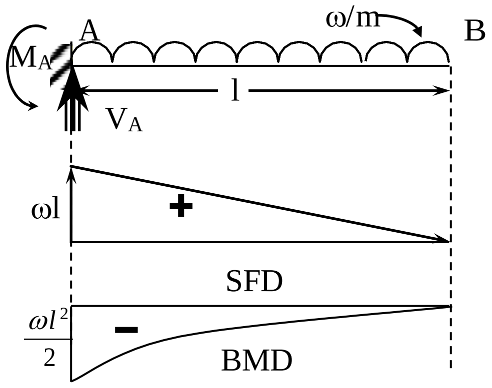


Fig. 3.34 Cantilever with UDL

Cantilever with Uniformly Varying Load (UVL)

Case (*i*)

The reactions can be obtained from the conditions of equilibrium.

$\sum V_A = 0;\ R_A = \frac{\omega l}{2}$

Taking moments about A,

$$M_A = -\left(\frac{\omega l}{2}\right) \times \left(\frac{l}{3}\right) = -\frac{\omega l^2}{6}$$

Shear Force Values

$V_B = 0$

$$V_A = \frac{\omega l}{2}$$

$$V_A = \frac{\omega l}{2} - \frac{\omega l}{2} = 0$$

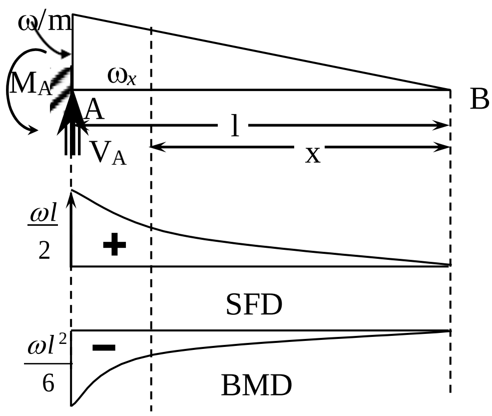


Fig. 3.35 Cantilever with UVL

Bending Moment Values

$M_B = 0$

$$M_A = -\left(\frac{\omega l}{2}\right) \times \left(\frac{l}{3}\right) = -\frac{\omega l^2}{6}$$

Consider a section at a distance x from free end and load intensity at that section ω_x is given by

$$\omega_x = \left(\frac{x}{l}\right)\omega$$

Shear Force at that section is given by

$$V_x = \frac{1}{2}\omega_x \times x = \left(\frac{\omega x^2}{2l}\right)$$

Bending Moment at that section is given by

$$M_x = -\left[\frac{1}{2}\omega_x \times x\right]\left(\frac{x}{3}\right) = -\left(\frac{\omega x^3}{6l}\right)$$

Case (*ii*)

The reactions can be obtained from the conditions of equilibrium.

$\sum V_A = 0$; $R_A = \dfrac{\omega l}{2}$

Taking moments about A,

$$M_A = \left(\frac{\omega}{2}l\right)\times\left(\frac{2l}{3}\right) = \frac{\omega l^2}{3}$$

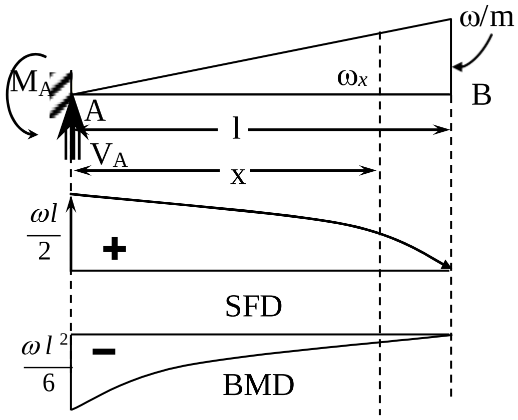


Fig. 3.36 Cantilever with UVL

Shear Force Values

$V_B = 0$

$$V_A = \frac{\omega l}{2}$$

$$V_A = \frac{\omega l}{2} - \frac{\omega l}{2} = 0$$

Bending Moment Values

$M_B = 0$

$$M_A = \left(\frac{\omega l}{2}\right)\times\left(\frac{2l}{3}\right) = \frac{\omega l^2}{3}$$

Consider a section at a distance x from free end and load intensity at that section ω_x is given by

$$\omega_x = \left(\frac{x}{l}\right)\omega$$

Shear Force at that section is given by

$$V_x = R_A - \frac{1}{2}\omega_x \times x = \left(\frac{\omega l}{2}\right) - \left(\frac{\omega x^2}{2l}\right)$$

Bending Moment at that section is given by

$$M_x = R_A \times x - \left[\frac{1}{2}\omega_x \times x\right]\left(\frac{x}{3}\right) - M_A = \left(\frac{\omega l}{2}\right)x - \left(\frac{\omega x^3}{6l}\right) - \left(\frac{\omega l^2}{3}\right)$$

Cantilever with Partial Uniformly Distributed Load (UDL)

The reactions can be obtained from the conditions of equilibrium.

$\sum V_A = 0$; $R_A = \omega b$ Taking moments about A,

$$M_A = -\omega b \times \left(a + \frac{b}{2}\right)$$

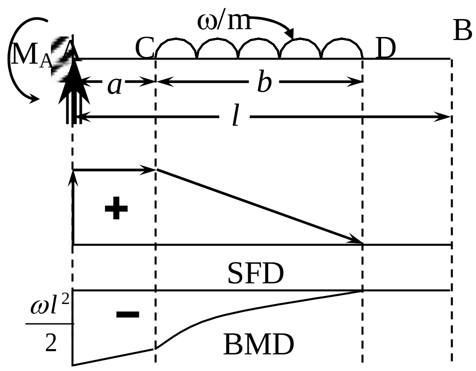


Fig. 3.37 Cantilever with Partial

Shear Force Values

$V_B = 0$

$V_D = 0$

$V_C = -\omega b$

$V_A = -\omega b$

$V_A = -\omega b + \omega b = 0$

Bending Moment Values

$M_B = 0$

$M_D = 0$

$$M_C = -\omega b \times \left(\frac{b}{2}\right) = -\frac{\omega b^2}{2}$$

$$M_A = -\omega b \times \left(a + \frac{b}{2}\right)$$

3.01. Draw the Shear Force and Bending Moment Diagram for a Cantilever beam subjected to concentrated loads as shown in Fig. 3.38.

From the conditions of equilibrium

$\sum V = 0$; $R_A = 10 + 20 + 30 = 60$ kN (↑)

$\sum M = 10 \text{ x } 6 + 20 \text{ x } 3 + 30 \text{ x } 2 = 180$ kN-m.

Shear Force Values at Salient Points

$V_D = 0 – 10 = -10$ kN

$V_C = -10 – 20 = -30$ kN

$V_B = -30 – 30 = -60$ kN

$V_A = -60 + 60 = 0$kN

Bending Moment Values at Salient Points

$M_D = 0$ kN-m

$M_C = -10 \text{ x } 3 = -30$ kN-m

$M_B = -10 \text{ x } 4 – 20 \text{ x } 1 = - 60$ kN-m

$M_A = -10 \text{ x } 6 – 20 \text{ x } 3 – 30 \text{ x } 2 = - 180$ kN-m

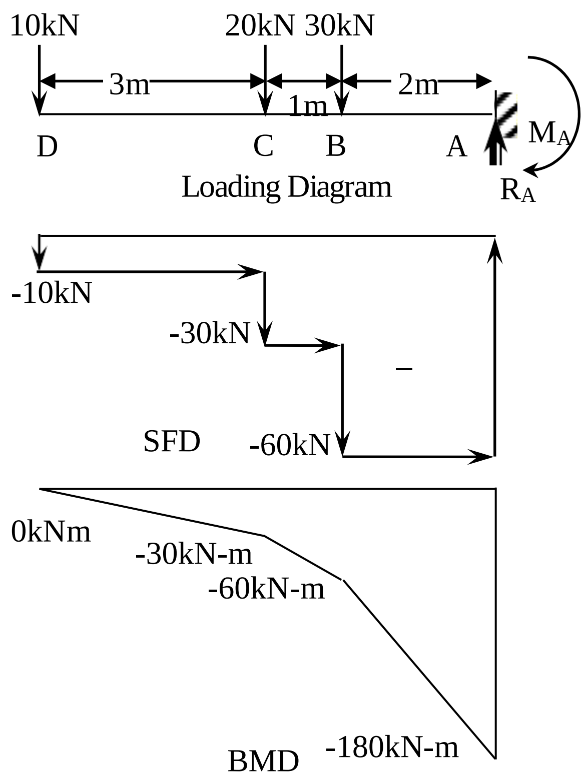


Fig.3.38 Cantilever

3.02. A cantilever beam is subjected to loads as shown in Fig. 3.39. Draw SFD and BMD.

From the conditions of equilibrium

$\sum V_A = 0$; $R_A = 10 + 30 + 20 \text{ x } 5 = 140$ kN (↑)

$\sum M_A = 30 \text{ x } 2 + 10 \text{ x } 3 + (20 \text{ x } 5)\left(\frac{5}{2}\right) + 40 = 380$ kN-m.

Shear Force Values at Salient Points

$V_D = 0$ kN

$V_C = 0 - 20 \text{ x } 2 = -40$ kN

$V_C = -40 - 10 = -50$ kN

$V_B = -50 - 20 \text{ x } 1 = -70$ kN

$V_B = -70 - 30 = -100$ kN

$V_A = -100 - 20 \text{ x } 2 = -140$ kN

$V_A = -140 + 140 = 0$kN

Bending Moment Values at Salient Points

As there is applied moment at section D, there will be two moments at that section and hence

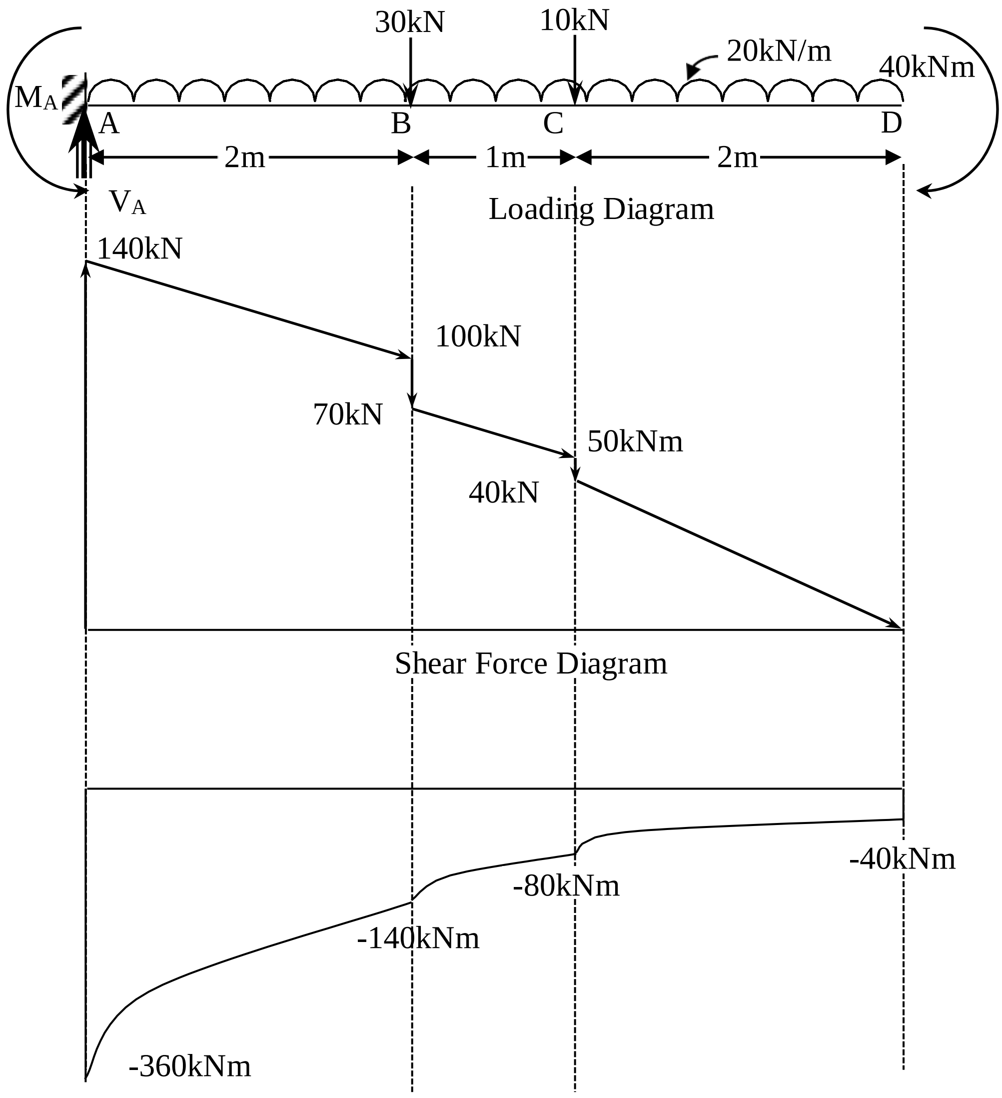
$M_{DR} = 0$

$M_{DL} = 0 - 40 = -40$kN-m

M_C = –20 x 2 x 1 – 40 = –80 kN-m

M_B = –20 x 3 x 1.5 – 10 x 1 – 40 = – 140 kN-m

M_A = –20 x 5 x 2.5 – 10 x 3 – 20 x 2– 40 = – 360 kN-m



Bending Moment Diagram

Fig. 3.39 BMD & SFD - Cantilever

3.03. Draw BMD and SFD for the cantilever beam shown in Fig. 3.40. Locate the point of contra flexure if any,

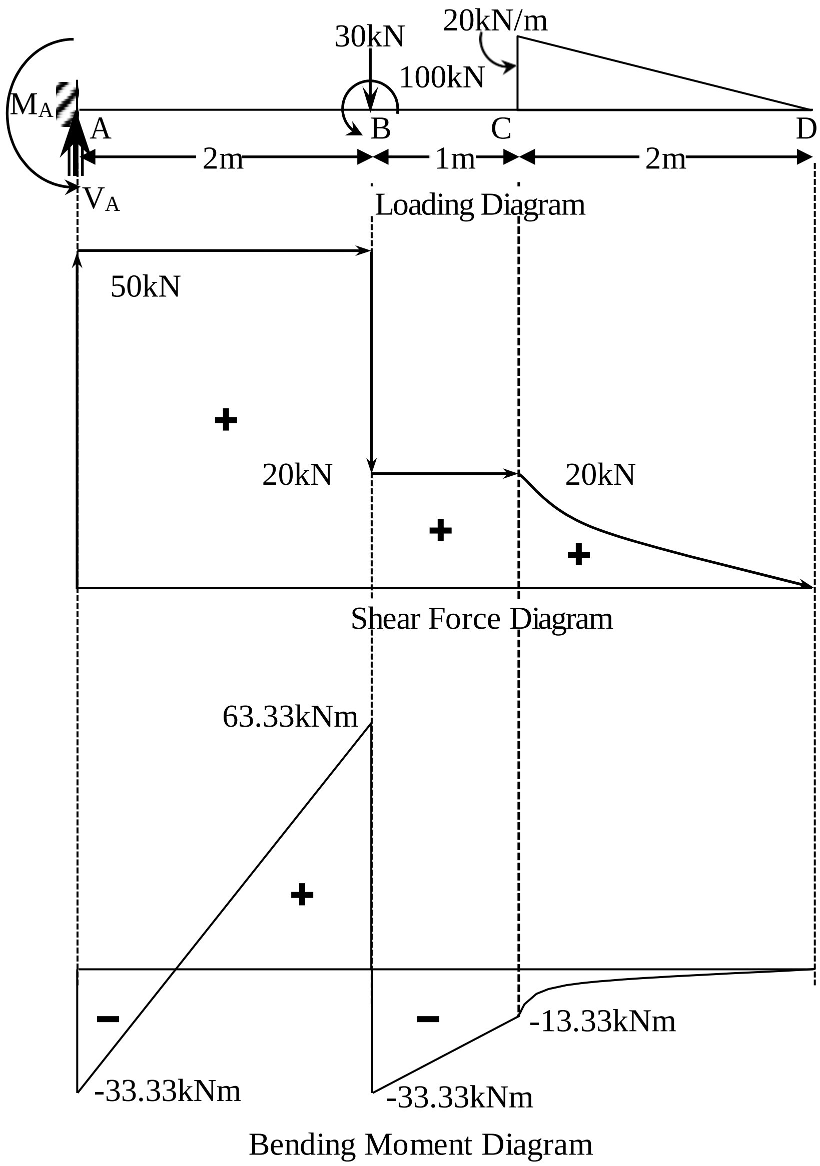


Fig. 3.40 BMD & SFD - Cantilever

From the conditions of equilibrium

$\Sigma V_A = 0$; $R_A = 30 + \left(\frac{1}{2}\right) \times 20 \times 2 = 50$ kN (↑)

$\Sigma M_A = 30 \times 2 + \left(\frac{1}{2}\right)(20 \times 2)\left(3 + \frac{2}{3}\right) - 100 = 33.33$ kN-m

Shear Force Values at Salient Points

$V_D = 0$ kN

$V_C = 0 - \left(\frac{1}{2}\right)(20 \times 2) = -20$ kN

$V_B = -20$ kN

$V_B = -20 - 30 = -50$ kN

$V_A = -50$ kN

$V_A = -50 + 50 = 0$kN

Bending Moment Values at Salient Points

As there is applied moment at section B, there will be two moments at that section and hence

$M_D = 0$ kN

$M_C = -\left(\frac{1}{2}\right)(20 \times 2)\left(\frac{2}{3}\right) = -13.33$ kN-m

$M_{BR} = -\left(\frac{1}{2}\right)(20 \times 2)\left(1+\frac{2}{3}\right) = -33.33$ kN-m

$M_{BL} = -33.33 + 100 = +66.67$kN-m

$M_A = -\left(\frac{1}{2}\right)(20 \times 2)\left(3+\frac{2}{3}\right) - 30 \times 2 + 100 = -33.33$ kN-m

Points of contraflexure:

$\frac{x}{33.33} = \frac{(2-x)}{66.67}$ or $x = 0.67$m

It lies at 0.67m and 2m right of the left support.

Bracket Connections

There can be following types of bracket connections which can be converted to load

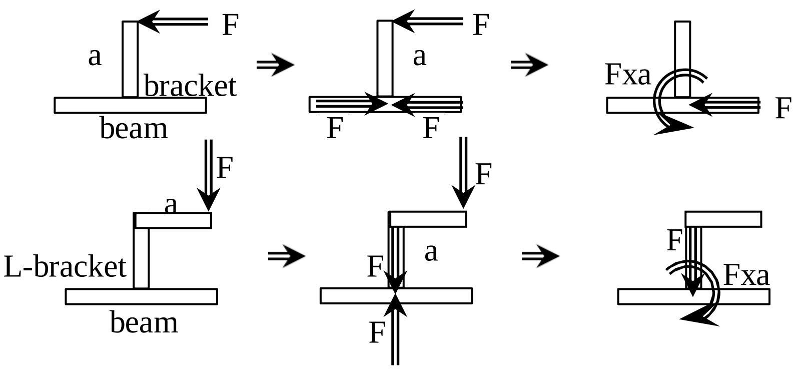


Fig.3.41 Bracket Connections

and moment.

The types of brackets are vertical and L bracket as shown in Fig. 3.41. Apply two equal, opposite and collinear forces at the joint where the load gets transferred to the beam. The two forces (F) acting equal and opposite separated by a distance will form a couple equal to the product of Force and the distance between the forces along with the remaining Force.

3.04. An overhanging beam ABC is loaded as shown in Fig. 3.42. Draw the shear force and bending moment diagrams. Also locate point of contraflexure. Determine maximum +ve and —ve bending moments. (Jan- 06)

The reactions can be obtained from the conditions of equilibrium.

$\sum V_A = 0;\ R_A + R_B = 2\times6 + 2 = 14\text{kN}$

Taking moments about A,

$$\Sigma M_A = 0; \qquad 4R_B = (2\times6)\left(\frac{6}{2}\right) + 2\times6 \text{ or } R_B = \frac{48}{4} = 12\text{kN}$$

Similarly taking moments about B,

$$\Sigma M_B = 0;\ 4R_B + 2\times2 + (2\times2)\left(\frac{2}{2}\right) = (2\times4)\left(\frac{4}{-}\right) \text{ or } R_A = \frac{8}{4} = 2\text{kN}$$

Check

Substituting in Eq. 01, we have $R_A + R_B = 2 + 12 = 14$ kN (O.K.)

Zero Shear Force

Consider a section at a distance x where Shear Force is zero as shown in Fig.3,42,

From similar triangles, we have

$$\frac{2}{x} = \frac{6}{(4-x)}$$

$$x = 1\text{m}$$

Bending Moment Values

$M_A = 0$

$$M_B = -2\times2 - 2\times2\times\left(\frac{2}{2}\right) = -8\text{kN}$$ (Negative because Sagging)

$M_C = 0$

Bending Moment at zero Shear Force will be either Maximum or Minimum.

$$M_x = 2x - \frac{2\times x^2}{2} = 2x - x^2 = 1\text{kNm}$$

Maximum positive BM is 1kNm at 1 m to right of left support and negative BM is 8kNm at right support.

Point of Contraflexure: Bending Moment equation at section y is

$$M_y = 2y - \frac{2\times y^2}{2} = 2y - y^2 \Rightarrow 0 \text{ or } y = 2\text{m}$$

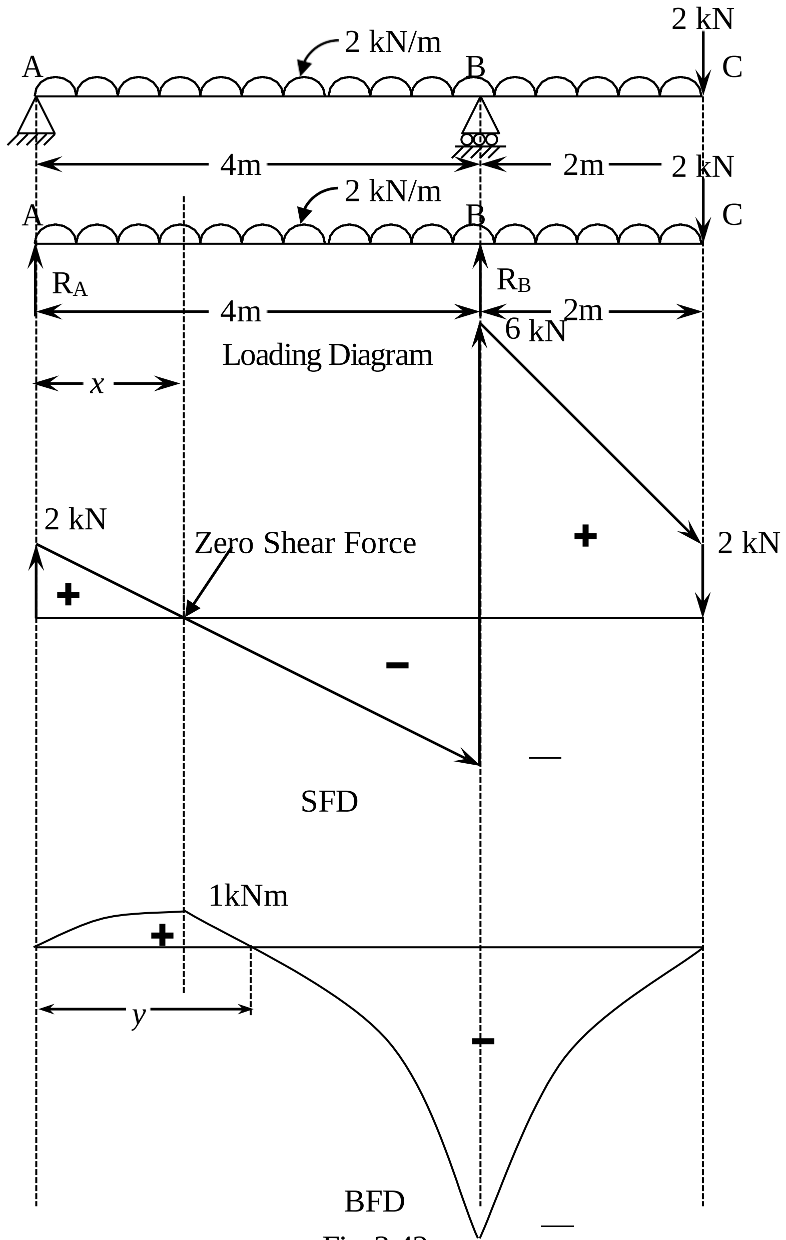


Fig. 3.42

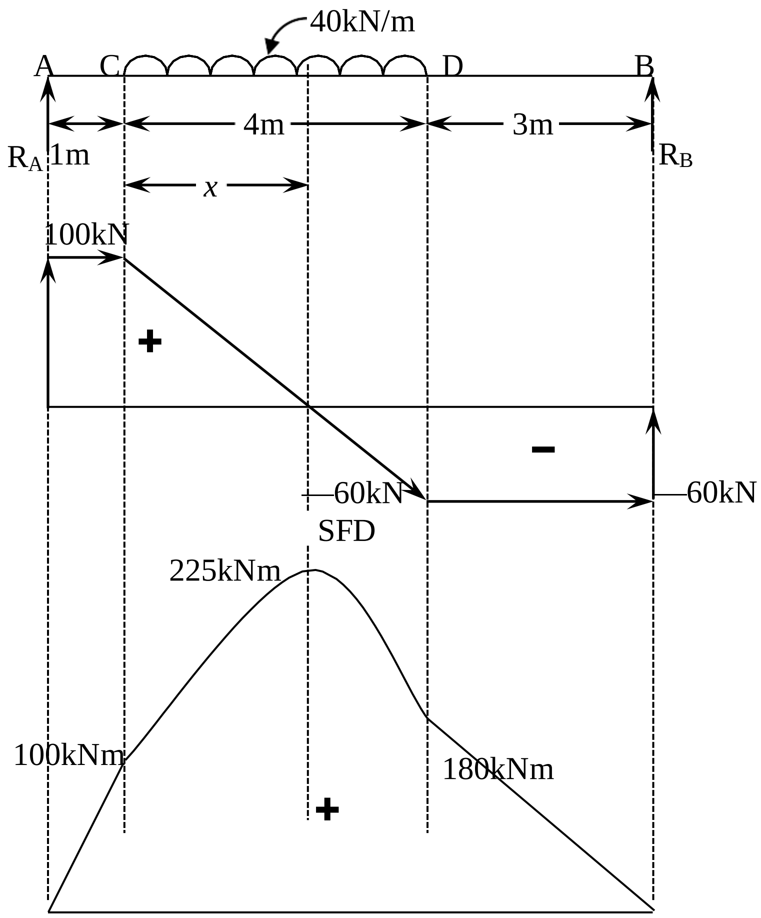
3.05. Draw the Shear Force and Bending Moment Diagram for the loaded beam shown in Fig. 3.43. Find the Maximum bending moment.

The reactions can be obtained from the conditions of equilibrium.

$\sum V_A = 0;\ R_A + R_B = 40\times4 = 160\text{kN}$ (01)

Taking moment about A,

$\Sigma M_A = 0; 8R_B = (40\times 4)\left(1 + \frac{4}{2}\right)$ or $R_B = \frac{480}{8} = 60\text{kN}$



BMD Fig. 3.43

The reactions can be obtained from the conditions of equilibrium.

$\sum V_A = 0;\ R_A + R_B = 40\times4 = 160\text{kN}$ (01)

Taking moment about A,

$\Sigma M_A = 0; 8R_B = (40\times 4)\left(1 + \frac{4}{2}\right) \text{ or } R_B = \frac{480}{8} = 60\text{kN}$

Similarly taking moment about B,

$\Sigma M_B = 0;\ 8R_A = (40 \times 4)\left(3 + \frac{4}{2}\right) \text{ or } R_A = \frac{800}{8} = 100\text{kN}$

Check

Substituting in Eq. 01, we have $R_A + R_B = 100 + 60 = 160$ kN (O.K.)

Zero Shear Force

Consider a section at a distance x where Shear Force is zero as shown in Fig. 3.43

From similar triangles, we have

$$\frac{100}{x} = \frac{60}{(4 - x)} \text{ or } x = 2.5\text{m}$$

$\boldsymbol{V_o = 1 + 2.5 = 3.5}$**m** from right support.

Bending Moment Values

$M_B = 0$

$M_D = 60\times3 = 180\text{kN}$

$M_C = 60\times7-(40\times4)\left(\frac{4}{2}\right)=100\text{kN}$

$M_A = 0$

Bending Moment at zero Shear Force will be either Maximum or Minimum.

$M_x = 100\times(1+x)-\frac{40\times x^2}{2}=100\times(1+x)-20\times x^2 = 225\text{kNm}$

3.06. Draw the Shear Force and Bending Moment Diagram for the loaded beam shown in Fig. 3.44. Also locate the Point of Contraflexure. Find and locate the Maximum +ve and —ve Bending Moments.

The reactions can be obtained from the conditions of equilibrium.

$\sum V_A = 0;\ R_C+R_D=40+20=60\text{kN}$ (01)

Taking moment about C,

$\Sigma M_C = 0; 4R_D + 2\times 40 = 20\times 6$ or $R_D = \frac{40}{4} = 10\text{kN}$

Similarly taking moments about D,

$\Sigma M_D = 0;\ 4R_C + 20\times 2 = 40\times 6$ or $R_C = \frac{200}{4} = 50\text{kN}$

Check

Substituting in Eq. 01, we have $R_C + R_D = 50 + 10 = 60$ kN (O.K.)

Zero Shear Force is at right support

Bending Moment Values

$M_B = 0$

$M_D = -20\times2 = -40\text{kN-m}$

$M_C = -40\times2 = -80\text{kNm}$

$M_A = 0$

Maximum Moments: Maximum negative BM is 80 kNm at the left support.

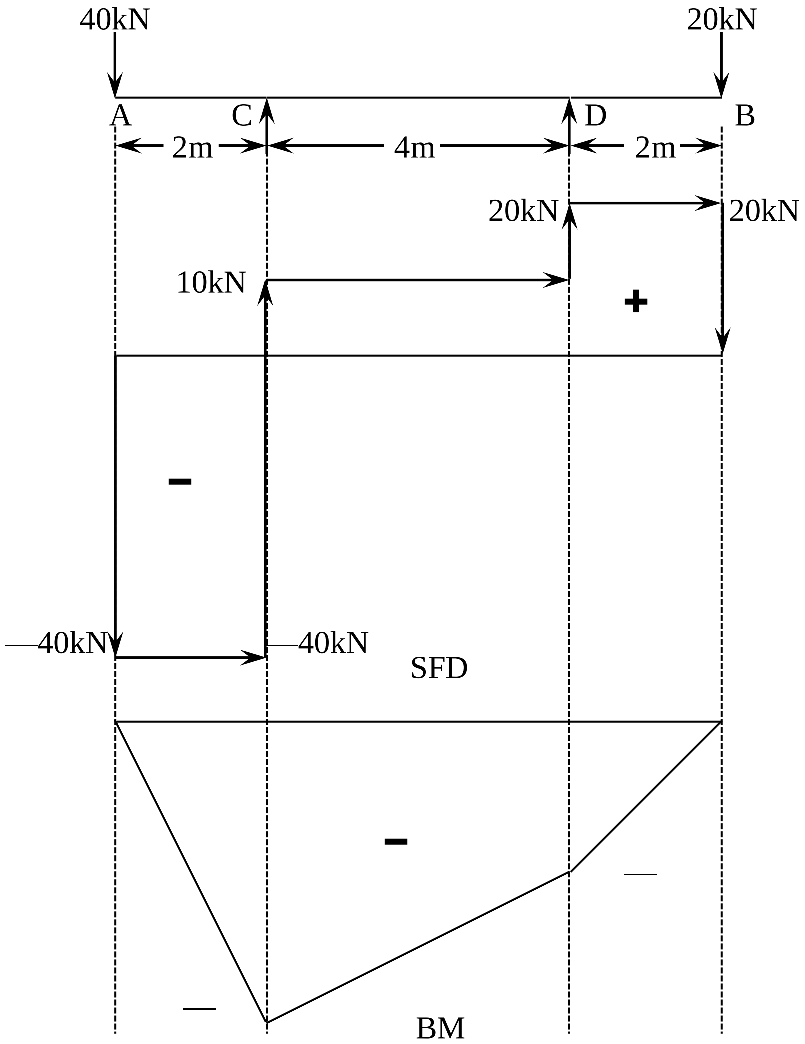


Fig. 3.44

3.07. Draw BMD and SFD for the loaded beam shown in Fig. 3.45. Also locate the Point of contraflexure and Maximum +ve and —ve Bending Moment

The reactions can be obtained from the conditions of equilibrium.

Taking moment about A,

$$\sum V_A = 0;\ R_A + R_B = 3 + 5 + 2\times 6 = 20\text{kN} \qquad (01)$$

$$\Sigma M_A = 0; 6R_B + 3\times 2 = (2\times 6)\left(\frac{6}{2}\right) + 5\times 8 \text{ or } R_B = \frac{70}{6} == 11.67\text{kN}$$

Similarly taking moment about B,

$$\Sigma M_B = 0;\ 6R_A + 5\times 2 = (2\times 6)\left(\frac{6}{2}\right) + 3\times 8 \text{ or } R_A = \frac{50}{6} = 8.33\text{kN}$$

Check: Substituting in Eq. 01, we have $R_A + R_B$ = 11.67 + 8.33 =20 kN (O.K.)

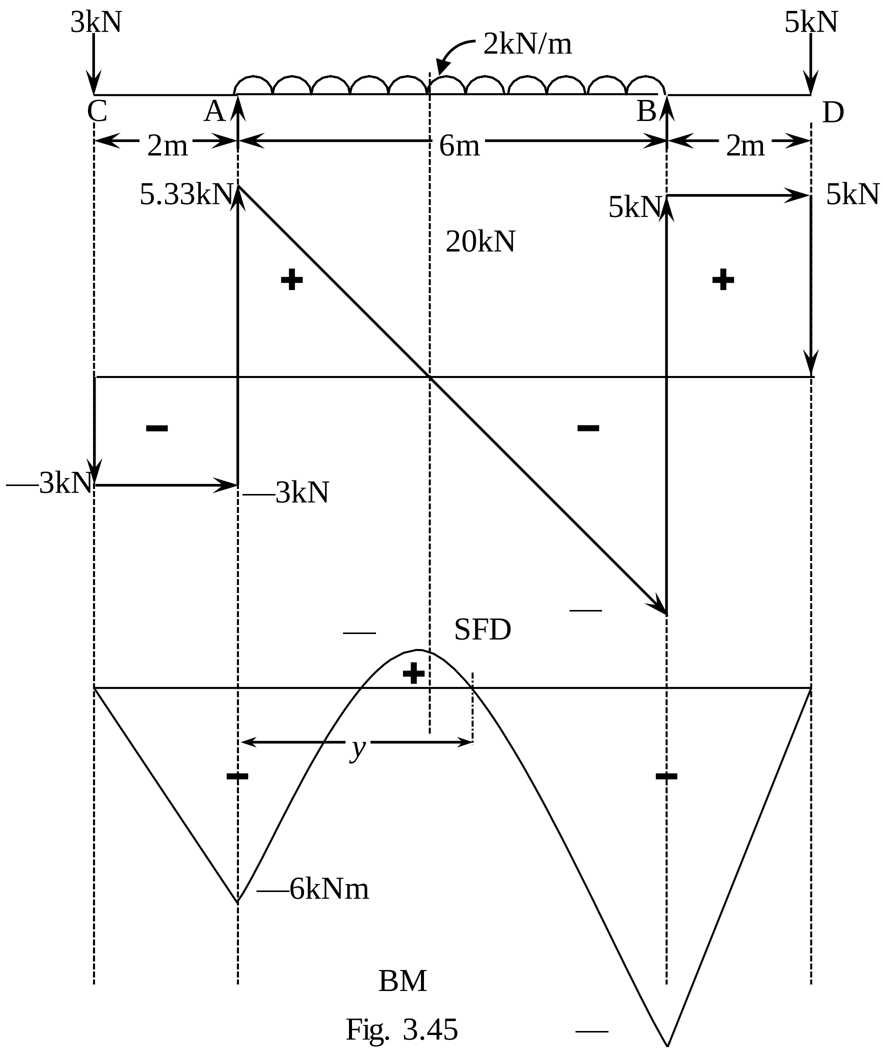


Fig. 3.45

Check: Substituting in Eq. 01, we have $R_A + R_B$ = 11.67 + 8.33 =20 kN (O.K.)

Zero Shear Force

Consider a section at a distance *x* where Shear Force is zero as shown in Fig. 3.45.

From similar triangles, we have

$$\frac{5.33}{x} = \frac{6.67}{(6-x)} \text{ or } x = 2.67\text{m}$$

Bending Moment Values

$M_D = 0$

$M_B = -5\times 2 = -10\text{kN}$

$M_A = -3\times 2 = -6\text{kN}$

$M_C = 0$

Bending Moment at zero Shear Force will be either Maximum or Minimum.

$$M_x = 8.33\times x - 3(2+x) - \frac{2\times x^2}{2} = 8.33\times x - 3(2+x) - \frac{2\times x^2}{2} = 1.11\text{kNm}$$

Points of Contraflexure:

Bending moment at section y from the left support is given by

$$M_y = 8.33y - 3\times(2+y) - \frac{2y^2}{2} \text{ or } y^2 - 5.33y + 6 = 0 \text{ and } y = 1.61\text{m and } 3.72\text{m}$$

Hence the points at 1.61m and 3.72m to right of left support.

3.08. Draw the BMD and SFD for the loaded beam shown in Fig. 3.46.

The reactions can be obtained from the conditions of equilibrium.

$\sum V_A = 0;\ R_A + R_B = 20\text{kN}$

Taking moment about A,

$$\Sigma M_A = 0; 3R_B = 20 \times 4 + 10$$

$$R_B = \frac{90}{3} = 30\text{kN}$$

Similarly taking moments about B,

$$M_B = 0;\ 3R_A + 10 + (20\times1) = 0$$

$$R_A = -\frac{30}{3} = -10\text{kN}$$

Check

Substituting in Eq. 01, we have $R_A + R_B = —10 + 30 = 20$ kN (O.K.)

Bending Moment Values

$M_D = 0$

$M_B = -20\times1 = -20\text{kNm}$ (Negative because Sagging)

$M_{C_R} = -20\times2 + 30\times1 = -10\text{kNm}$

$M_{C_L} = -10 - 10 = -20\text{kNm}$ or (By considering right side forces)

$M_{C_L} = -10\times2 = 20\text{kNm}$ (By considering left side forces)

$M_A = 0$

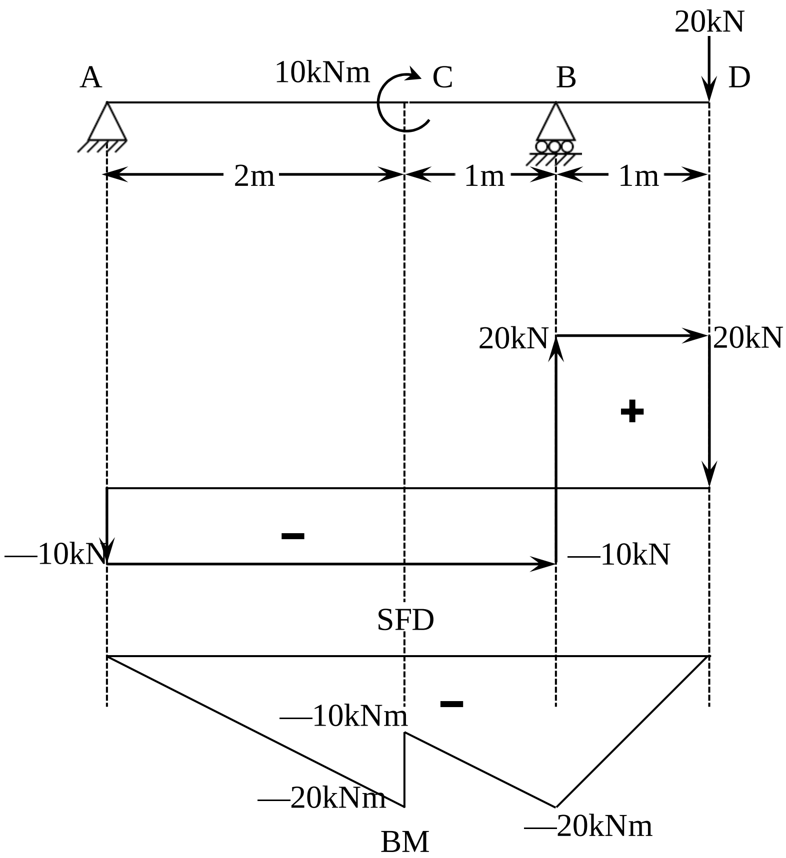


Fig. 3.46

An overhang beam ABC is loaded as shown in Fig. 3.47. Draw BMD and SFD.

The reactions can be obtained from the conditions of equilibrium.

$\sum V_A = 0;\ R_A + R_B = 4\times3+12 = 24\text{kN}$

Taking moment about A,

$\Sigma M_A = 0; 6R_B = 12 \times 9 + (4 \times 3)\left(3 + \frac{3}{2}\right) \text{ or } R_B = \frac{162}{6} = 27\text{kN}$

Similarly taking moments about B,

$M_B = 0;\ 6R_A \ +12\times3 = (4\times3)\left(\frac{3}{2}\right) \text{ or } R_A = -\frac{18}{6} = -3\text{kN}$

Check

Substituting in Eq. 01, we have $R_A + R_B = —3 + 27 = 24$ kN (O.K.)

Bending Moment Values

$M_D = 0$

$M_B = -12\times3 = -36\text{kNm}$ (Negative because Sagging)

$M_C = -3\times3 = -6\text{kNm}$

$M_A = 0$

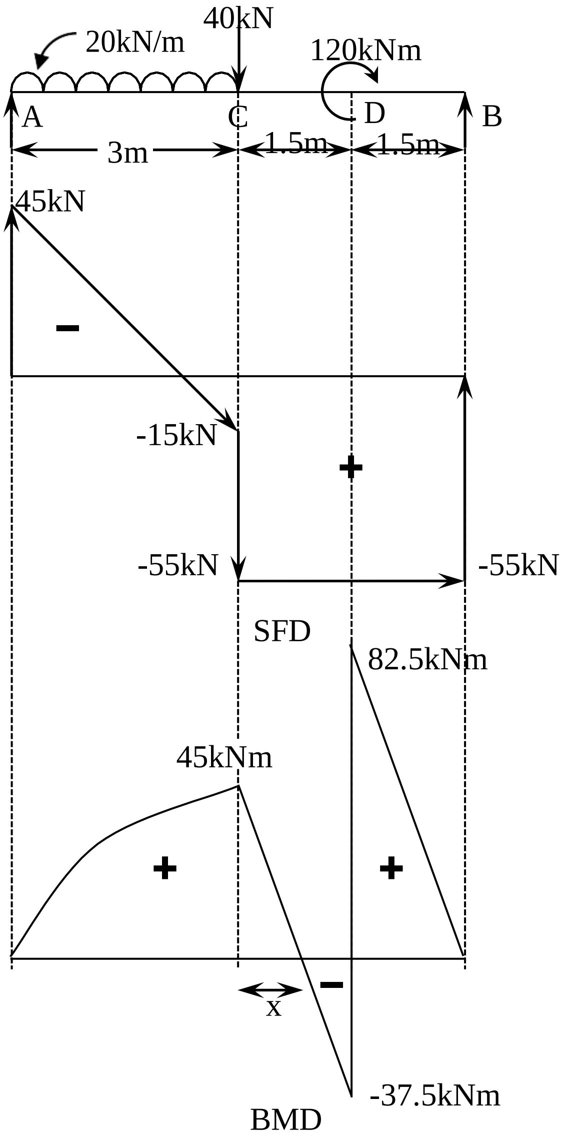


Fig. 3.48

3.09. Draw SFD and BMD for the beam shown in Fig. 3.48. Determine the maximum BM and its location. Locate the points of contraflexure. (July 02)

The reactions can be obtained from the conditions of equilibrium.

$\sum V_A = 0;\ R_A + R_B = 20\times3 + 40 = 100\text{kN}$

Taking moment about A,

$$\Sigma M_A = 0; 6R_B = (20\times3)\left(\frac{3}{2}\right) + 40\times3 + 120 \text{ or } R_B = \frac{330}{6} = 55\text{kN}$$

Similarly taking moments about B,

$$M_B = 0;\ 6R_A = 40\times3 + (20\times3)\left(3+\frac{3}{2}\right) - 120 \text{ or } R_A = \frac{270}{6} = 45\text{kN}$$

Check

Substituting in Eq. 01, we have $R_A + R_B = 45 + 55 = 100$ kN (O.K.)

Bending Moment Values

$M_B = 0$

$M_{D_R} = 55\times1.5 = 82.5\text{kNm}$

$M_{D_L} = 82.5 - 120 = -37.5\text{kNm}$ (By considering right side forces)

$M_{D_L} = 45\times4.5 - (20\times3)\left(1.5+\frac{3}{2}\right) - 40\times1.5 = -37.5\text{kNm}$ (By left side forces)

$M_C = 55\times3 - 120 = 45\text{kNm}$ (By considering right side forces)

$M_C = 45\times3 - (20\times3)\left(\frac{3}{2}\right) = 45\text{kNm}$ (By left side forces)

$M_A = 0$

Points of Contraflexure

Consider a section at a distance *x* where BM is changing its sign as shown in Fig. 3.49. From similar triangles, we have

$$\frac{45}{x} = \frac{37.5}{(1.5-x)}$$

$x = 0.818\text{m}$

The Points of contraflexure are located at 3.818m and 4.5m from the left support.

3.10. A beam ABCDE is 12m long simply supported at points B and D. Spans AB=DE=2m is overhanging. BC=CD=4m. The beam supports a udl of 10kN/m over AB and 20kN/m over CD. In addition it also supports concentrated load of 10kN at E and a clockwise moment of 16kNm at point C. Sketch BMD and SFD. (Aug 05)

The reactions can be obtained from the conditions of equilibrium.

$\sum V_A = 0;\ R_B + R_D = 10\times2 + 20\times4 + 10 = 110\text{kN}$ (01)

Taking moment about B,

$\Sigma M_B = 0; 8R_D + (10\times2)\left(\frac{2}{2}\right) = 10\times10 + (20\times4)\left(4+\frac{4}{2}\right) + 16 \text{ or } R_D = \frac{576}{8} = 72\text{kN}$

Similarly taking moment about D,

$\Sigma M_D = 0;\ 8R_B + 10\times2 + 16 = (10\times2)\left(8+\frac{2}{2}\right) + (20\times4)\left(\frac{4}{2}\right) \text{ or } R_B = \frac{304}{8} = 38\text{kN}$

Check

Substituting in Eq. 01, we have $R_B + R_D = 38 + 72 = 110$ kN (O.K.)

Zero Shear Force

Consider a section at a distance x where Shear Force is zero as shown in Fig. 3.50.

From similar triangles, we have

$$\frac{12}{x} = \frac{68}{(4-x)} \quad \text{or } x = 0.6\text{m}$$

Bending Moment Values

$M_E = 0$

$M_D = -10\times 2 = -20\text{kN}$

$M_{C_R} = 72 \times 4 - 10 \times 6 - (20 \times 4)\left(\frac{4}{2}\right) = 68\text{kNm}$

$M_{C_L} = 68 - 16 = 52\text{kNm}$ (From right side forces)

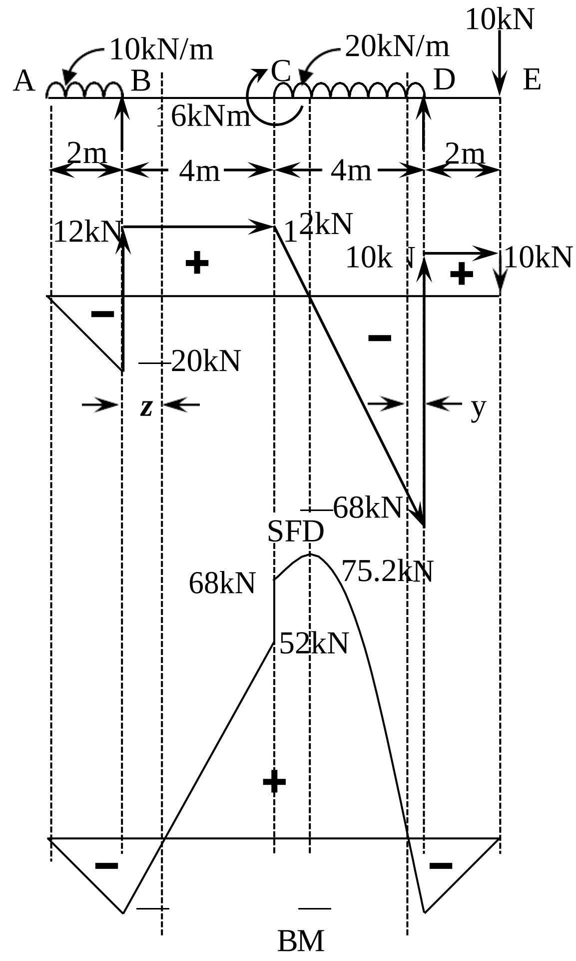


Fig. 3.49

$M_{C_L} = 38\times 4 - (10\times 2)\left(4 + \frac{2}{2}\right) = 52\text{kNm}$ (From left side forces)

$M_B = -(10 \times 2)\left(\frac{2}{2}\right) = -20\text{kNm}$

$M_A = 0$

Bending Moment at zero Shear Force will be either Maximum or Minimum.

$$M_x = 72\times(4-x)-10(2+4-x)-\frac{20\times(4-x)^2}{2}$$
$$= 72\times(4-0.6)-10(2+4-0.6)-10(4-0.6)^2 = 75.2\text{kNm}$$

Point of Contraflexures

Consider a section at a distance *z* where Bending Moment is zero as shown in Fig. 3.49. From similar triangles, we have

$$\frac{20}{z}=\frac{52}{(4-z)} \text{ and } z = 1.1\text{m}$$

Bending Moment at Section *y* from point D is zero and can be written as

$$M_y = 72\times y-10(2+y)-\frac{20\times y^2}{2} = 0$$
$$= 72\times y-10(2+y)-10\times y^2 = 62y-10y^2-20 = 0 \text{ and } y = 0.341\text{m}$$

3.11. Draw the Shear Force and Bending Moment Diagrams for the beam shown in Fig. 3.50. Locate the point of contraflexure if any. (Feb 04)

The reactions can be obtained from the conditions of equilibrium.

$\sum V_A = 0;\ R_A + R_D = (10\times5) + 80 + 80 + (16\times 2.5) = 250\text{kN}$

Taking moment about A,

$$\Sigma M_A = 0; 12.5R_D = (10\times5)\left(\frac{5}{2}\right)+80\times5+80\times7.5+(16\times2.5)\left(12.5+\frac{2.5}{2}\right)$$

$$R_D = \frac{1675}{12.5} = 134\text{kN}$$

Similarly taking moments about B,

$$\Sigma M_D = 0; 12.5R_A + (16\times2.5)\left(\frac{2.5}{2}\right) = (10\times5)\left(7.5+\frac{5}{2}\right)+80\times7.5+80\times5 =$$

$$R_A = \frac{1450}{12.5} = 116\text{kN}$$

Check

Substituting in Eq. 01, we have $R_A + R_B = 116 + 134 = 250$ kN (O.K.)

Bending Moment Values

$M_E = 0$

$$M_D = -(16\times2.5)\left(\frac{2.5}{2}\right) = -50\text{kNm}$$

$$M_C = 134\times5-(16\times2.5)\left(5+\frac{2.5}{2}\right) = 425\text{kNm}$$

$$M_B = 116\times5-(10\times5)\left(\frac{5}{2}\right) = 455\text{kNm}$$

$M_A = 0$

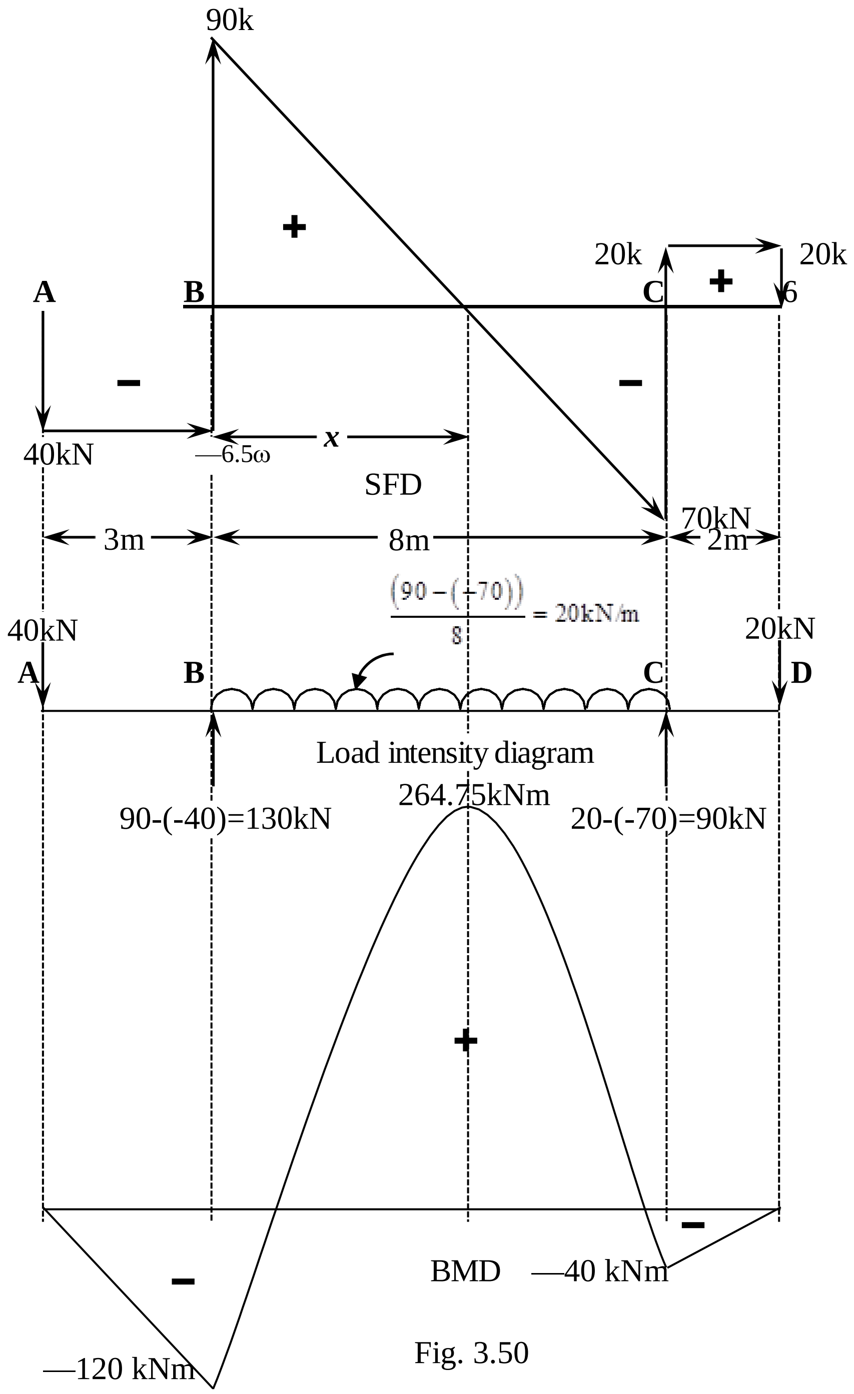


Fig. 3.50

Point of Contraflexure

Consider a section at a distance y from the right support where Bending Moment is zero as shown in Fig. From similar triangles, we have

$$\frac{50}{y} = \frac{425}{(5-y)} \text{ and } z = 0.526\text{m}$$

3.12. From the given shear force diagram shown in the Fig. 3.50, develop the load intensity diagram and draw the corresponding bending moment diagram indicating the salient features. (Jan 08)

The vertical lines in Shear force diagram represent vertical load, horizontal lines indicate generally no load portion, inclined line represents udl and parabola indicates uniformly varying load.

To generate load intensity diagram, the computations are shown in Fig. 3.50. The vertical line from the horizontal line below the line indicates negative value and vice versa. To check whether the applied moments are there in the loading diagram, we can take algebraic sum of moments of all the loads about any point and if there is a residue from the equation it indicates the applied moment in the opposite rotation to be applied anywhere on the beam.

Check

Taking Moments about B, we have

$$\Sigma M_B = 0; 40 \times 3 + 90 \times 8 - 20 \times 10 - (20 \times 8)\left(\frac{8}{2}\right) = 0$$

Note: Hence there is no applied moment or couple and if there is any residue from the equation like $+M$ kNm then there is an applied moment of M kNm clockwise and vice versa.

Bending Moment Values

$M_D = 0$

$M_C = -20 \times 2 = -40$ kNm (Negative due to hogging moment)

$M_B = -40 \times 3 = -120$ kNm (Negative due to hogging moment)

$M_A = 0$

Maximum Bending Moment occurs at zero shear force which is located at a distance x from the left support as shown in Fig. From similar triangles, we have

$$\frac{90}{x} = \frac{70}{(8-x)} \text{ or } x = 4.5\text{m}$$

Maximum Bending Moment at the section x is

$$M_x = 130x - 40\times(3+x) - \frac{20x^2}{2} = 130x - 40\times(3+x) - x^2$$

$$= 130 \times 4.5 - 40\times(3+4.5) - 4.5^2 = 264.75\text{kNm}$$

3.13. A beam 6m long rests on two supports with equal overhangs on either side and carries a uniformly distributed load of 30kN/m over the entire length of the beam as shown in Fig. 3.51. Calculate the overhangs if the maximum positive and negative bending moments are to be same. Draw the SFD and BMD and locate the salient points. (Jan 07)

The reactions can be obtained from the conditions of equilibrium.

As the loading is symmetrical $R_A = R_B$ and hence

$\sum V_A = 0$; $R_B + R_C = 2\,R_B = 2R_C = 30 \text{ x } (6+2a)$

$$R_B = R_C = \frac{30\times 6}{2} = 90\text{kN}$$

Bending Moment at any section x from the left end is given by

$$M_x = 90\,(x-a) - \frac{30x^2}{2} \text{ or } 90(x-a) - 15x^2 \qquad 01$$

From the given problem, maximum positive and negative bending moments are to be same, which occurs at zero shear force sections. From the above loading diagram, it can be seen that the zero shear force occurs at support and at centre (as the loading

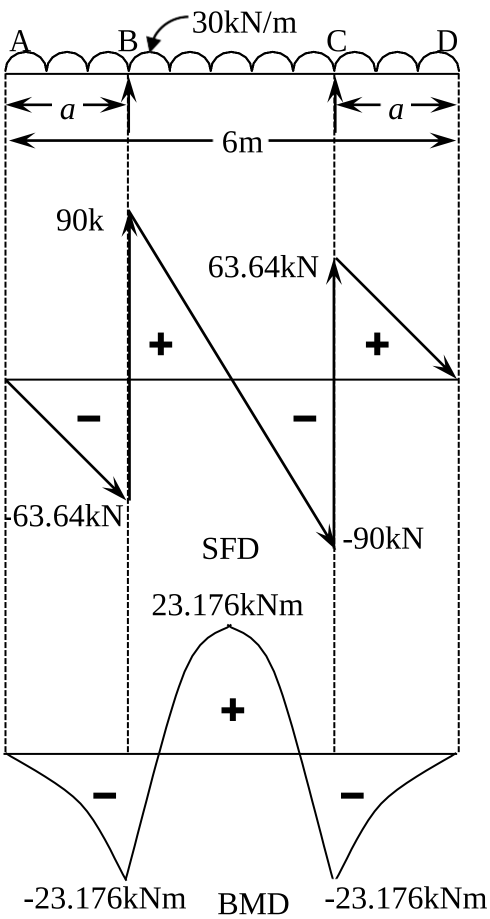


Fig. 3.51

is symmetrical). Hence substituting $x = a$ and 3, we get maximum +ve and —ve Bending Moment.

$$M_B = -15a^2$$

$$M_E = 90(3-a) - 15(3)^2 = 90(3-a) - 135$$

Equating the absolute values of above two equations, we have

$15a^2 = 90(3-a) - 135$ or $a^2 + 6a - 9 = 0$ and $a = 1.243\text{m}$

Bending Moment Values

$M_D = 0$

$$M_C = -\frac{30 \times 1.243^2}{2} = -23.176\text{kNm}$$

$$M_B = -\frac{30 \times 1.243^2}{2} = -23.176\text{kNm}$$

$M_A = 0$

$$M_E = 90(3-1.243) - \frac{30 \times 1.243^2}{2} = 23.176\text{kNm}$$

Points of Contraflexure:

$$M_x = 90(x-1.243) - 15x^2 = 6(x-1.243) - x^2 \Rightarrow 0 \text{ or } x = 1.76\text{m and } 4.24\text{m}$$

The points of contraflexure are at 1.76m and 4.24m from left end.

3.14. Draw the Shear Force and Bending Moment Diagram for a simply supported beam subjected to uniformly varying load shown in Fig. 3.52.

The trapezoidal load can be split into udl and uvl (triangular load) as shown in Fig. 3.43.

$\sum V_A = 0;\ R_A + R_B = (15 \times 6) + \left(\frac{1}{2}\right)(10 \times 6) = 120\text{kN}$ 01

Taking moment about A,

$$\Sigma M_A = 0; 6R_B = (15 \times 6)\left(\frac{6}{2}\right) + \left(\frac{1}{2}\right)(10 \times 6)\left(\frac{2}{3} \times 6\right) \text{ or } R_B = \frac{390}{6} = 65\text{kN}$$

Similarly taking moments about B,

$$\Sigma M_B = 0; 6R_A = (15 \times 6)\left(\frac{6}{2}\right) + \left(\frac{1}{-}\right)(10 \times 6)\left(\frac{6}{-}\right) + 80 \times 7.5 + 80 \times 5 \text{ or } R_A = \frac{330}{6} = 55\text{kN}$$

Check

Substituting in Eq. 01, we have $R_A + R_B = 55 + 65 = 120$ kN (O.K.)

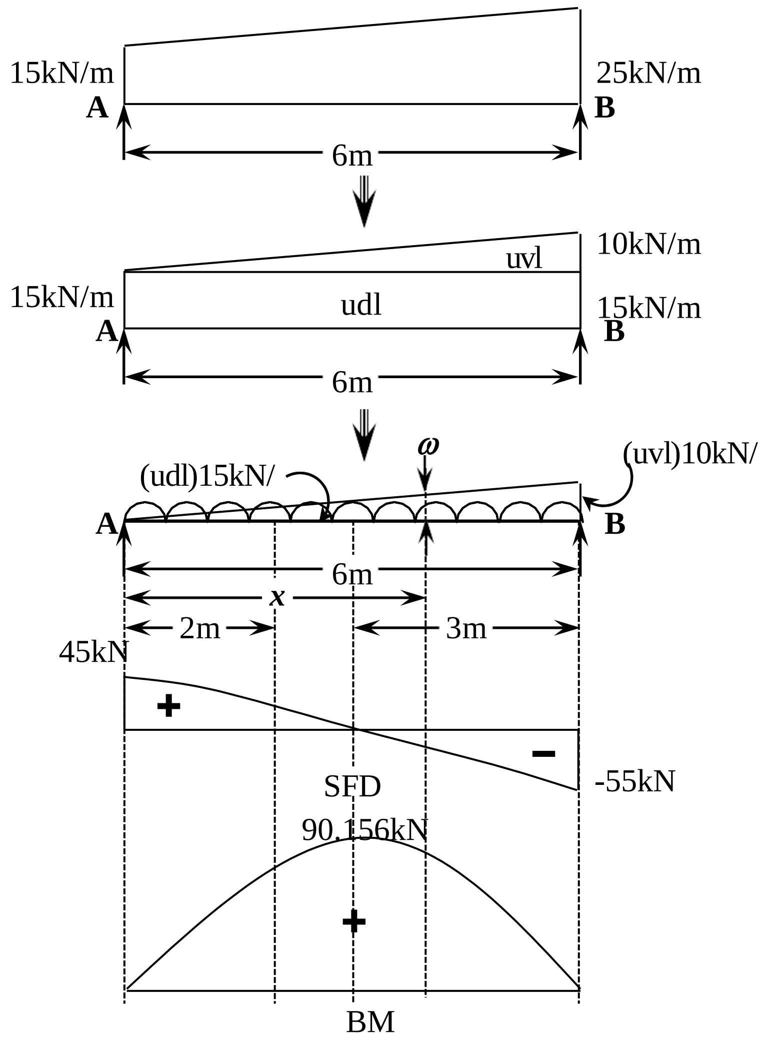


Fig.

Shear Force Equation at any section *x* from left support

Consider a section x at a distance x from the left support as shown.

The intensity of uvl at x is given by

$$\omega_x = \left(\frac{10 \times x}{6}\right) = 1.67x \text{ kN/m}$$

$$V_x = 55 - 15x - \frac{1.67x^2}{2} = 55 - 15x - \frac{5}{6}x^2 \text{ kN}$$

At x = 2m, $V_2 = 55 - 15 \times 2 - \frac{5}{6} \times 2^2 = 21.67\text{kN}$

At x = 3m, $V_3 = 55 - 15 \times 3 - \frac{5}{6} \times 3^2 = 2.5\text{kN}$

At x = 5m, $V_5 = 55 - 15 \times 5 - \frac{5}{6} \times 5^2 = -40.83\text{kN}$

Zero Shear Force = $V_o = 55 - 15 \times x - \frac{5}{6} \times x^2 = 0$ solving we get, $x = 3.124\text{m}$

Bending Moment Values

Bending Moment Equation at any section *x* from left support

Consider a section *x* at a distance *x* from the left support as shown.

$$M_x = 55x - \frac{15x^2}{2} - \left(\frac{1.67x^2}{2}\right)\left(\frac{x}{3}\right) = 55x - 7.5x^2 - \frac{5}{18}x^3 \text{ kNm}$$

$$M_x = 55\times -7.5x^2 - \frac{5}{18}x^3$$

$M_B = 0$

$M_A = 0$

Maximum Bending Moment occurs at SF = 0, i.e. $x = 3.124$m

$$M_x = 55\times 3.124 - 7.5\times 3.124^2 - \left(\frac{5}{18}\right)\times 3.124^3 = 90.156\text{kNm}$$

3.15. A beam ABCD 20m long is loaded as shown in Fig. 3.53. The beam is supported at B and C with a overhang of 2m to the left of B and a overhang of *a*m to the right of support C. Determine the value of *a* if the midpoint of the beam is point of inflexion and for this alignment plot BM and SF diagrams indicating the important values.

The reactions can be obtained from the conditions of equilibrium.

$\sum V_A = 0;\ R_B + R_C = 5\omega + \omega\times 20 = 25\omega\text{kN}$ (01)

Taking moment about B,

$$\Sigma M_B = 0; (18-a)R_C + (5\omega)\times 2 + \left(\frac{\omega\times 2^2}{2}\right) = \left(\frac{\omega\times(20-2)^2}{2}\right)$$

$$(18-a)R_C = 150\omega \text{ or } R_C = \frac{150\omega}{(18-a)}$$

Similarly taking moment about C,

$$\Sigma M_C = 0; (18-a)R_B + \left(\frac{\omega a^2}{2}\right) = (5\omega)(20-a) + \left(\frac{\omega\times(20-a)^2}{2}\right)$$

$$(18-a)R_B = 300\omega - 25a\omega \text{ or } R_B = \frac{\omega(300-25a)}{(18-a)}$$

Check

Substituting in Eq. 01, we have

$$R_B + R_C = \frac{150\omega}{(18-a)} + \frac{\omega(300-25a)}{(18-a)} = 25\omega \text{ (O.K.)}$$

Point of contraflexure

Consider a section at a distance x from left support as shown in Fig. 3.53. Bending moment at this section is given by

$$M_x = R_B \times (x-2) - 5\omega \times x - \frac{\omega x^2}{2} = \left[\frac{\omega(300-25a)}{(18-a)}\right] \times (x-2) - 5\omega \times x - \frac{\omega x^2}{2}$$

From the given data, this is zero at x = 10m. Hence

$$\left[\frac{\omega(300-25a)}{(18-a)}\right] \times (x-2) - 5\omega \times x - \frac{\omega x^2}{2} = 0$$

$$\left[\frac{(300-25a)}{(18-a)}\right] \times 8 - 5 \times 10 - \frac{10^2}{2} = 0$$

$$\left[\frac{(300-25a)}{(18-a)}\right] = 12.5$$

$300 - 25a = 225 - 12.5a$ or $a = 6$m

$$R_B = \frac{\omega(300-25a)}{(18-a)} = \frac{\omega(300-25\times 6)}{(18-6)} = 12.5\omega$$

$$R_C = \frac{150\omega}{(18-a)} = \frac{150\omega}{(18-6)} = 12.5\omega$$

Zero Shear Force

Consider a section at a distance y where Shear Force is zero as shown in Fig. 3.53. From similar triangles, we have

$$\frac{5.5}{y} = \frac{6.5}{(12-y)} \text{ or } y = 5.5\text{m}$$

Bending Moment Values

$\boldsymbol{M_D = 0}$

$$M_C = -\omega \times \frac{6^2}{2} = -18\omega$$

$$M_B = -5\omega \times 2 - \omega \times \frac{2^2}{2} = -12\omega$$

$\boldsymbol{M_A = 0}$

$$M_E = 12.5\omega \times 5.5 - 5\omega \times (5.5+2) - \frac{\omega(5.5+2)^2}{2} = 3.125\omega$$

Another point of contraflexure is

$$M_x = \left[\frac{\omega(300-25\times 6)}{(18-6)}\right] \times (6-2) - 5\omega \times 6 - \frac{\omega 6^2}{2}$$

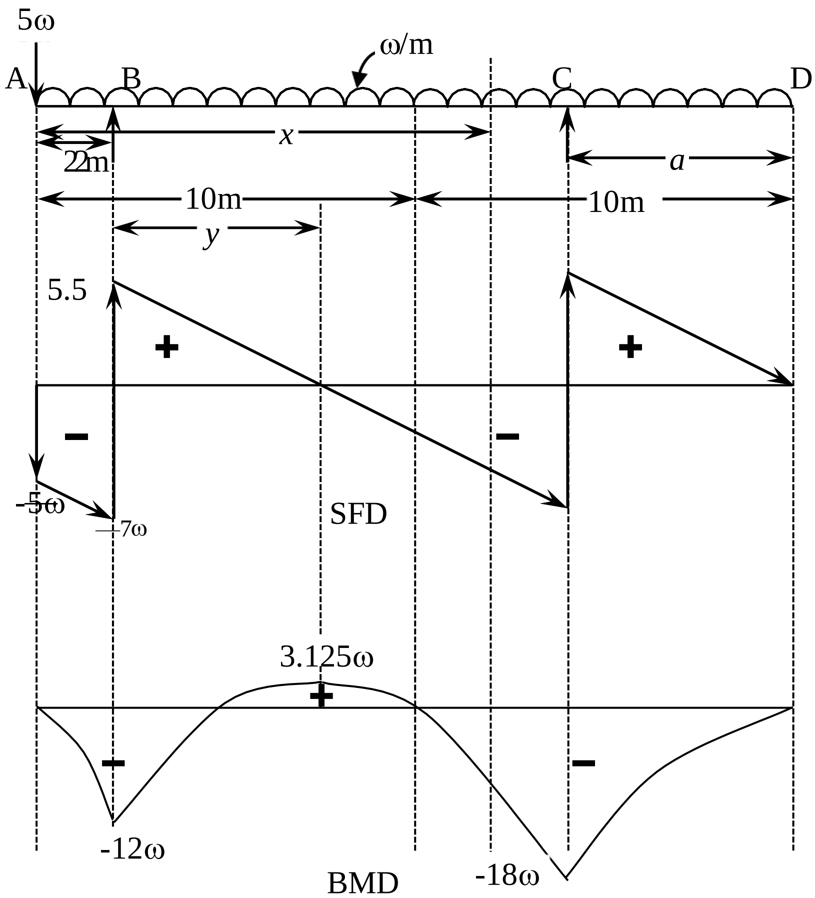


Fig. 3.53

3.16 For the beam AC shown in Fig. 3.54, determine the magnitude of the load P acting at C such that the reaction at supports A and B are equal and hence draw the Shear force and Bending moment diagram. Locate points of contraflexure. (July 08)

The reactions can be obtained from the conditions of equilibrium.

$\sum V_A = 0;\ R_A + R_B = 45 \times 4 + P$ 01

From the given data, $R_A = R_B$ and substituting in Eq. 01, $2R_A = 2R_B = 180 + P$

Taking moment about A,

$$\Sigma M_A = 0; 6R_B = 7P + (45 \times 4)\left(\frac{4}{2}\right) + 30 \text{ or } 6R_B = 7P + 390$$

Substituting from Eq. 01,

$$3(180 + P) = 7P + 390 \text{ or } P = 37.5\text{kN}$$

Check

Similarly taking moments about B,

$$\Sigma M_B = 0; 6R_A + P \times 1 + 30 = (45 \times 4)\left(2 + \frac{4}{2}\right)$$

$$6R_A = 690 - P$$

Substituting from Eq. 01, $3(180 + P) = 690 - P$ or $P = 37.5\text{kN}$

Hence O.K.

$2R_A = 2R_B = 180 + 37.5 = 217.5$kN

$R_A = R_B = 108.75$kN

Zero Shear Force

Consider a section at a distance x where Shear Force is zero as shown in Fig. 3.54.

From similar triangles, we have

$$\frac{108.75}{x} = \frac{71.25}{(4-x)} \text{ or } x = 2.417\text{m}$$

Bending Moment Values

$M_C = 0$

$M_B = -37.5 \times 1 = -37.5$kNm (Negative because Sagging)

$M_{D_R} = 108.75 \times 2 - 37.5 \times 3 = 105$kNm

$M_{D_L} = 108.75 \times 4 - (45 \times 4)\left(\frac{4}{2}\right) = 75$kNm (From left side forces)

$M_{D_L} = 105 - 30 = 75$kNm (From Right side forces)

$M_A = 0$

Maximum Bending moment occurs at zero shear force. i.e. at $x = 2.417$

$$M_x = 108.75 \times x - \frac{45 \times x^2}{2} = 108.75 \times 2.417 - \frac{45 \times 2.417^2}{2} = 131.41\text{kNm}$$

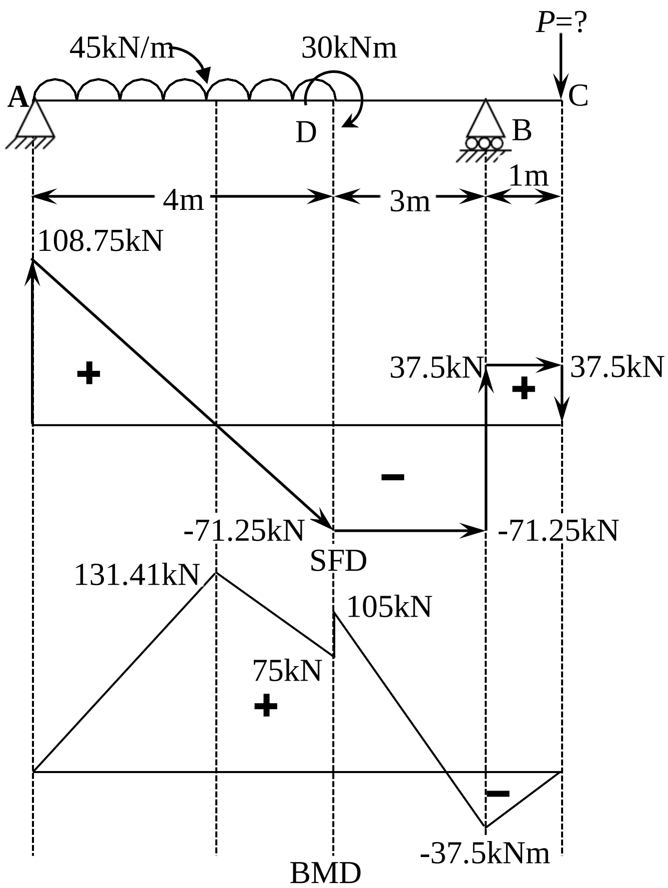


Fig. 3.54

3.16. Draw the bending moment and shear force diagrams for a prismatic simply supported beam of length *L*, subjected to a clockwise moment *M* at the centre of the beam and a uniformly distributed load of intensity *q* per unit length acting over the entire span. (Jan 09)

The reactions can be obtained from the conditions of equilibrium.

$\sum V_A = 0;\ R_A + R_B = q \times L$ kN (01)

Taking moment about A,

$$\Sigma M_A = 0;\ R_B \times L + M = \frac{q \times L^2}{2}$$

$$R_B = \frac{q \times L}{2} - \frac{M}{L}$$

Similarly taking moment about B,

$$\Sigma M_B = 0;\ R_A \times L = \frac{q \times L^2}{2} + M$$

$$R_A = \frac{q \times L}{2} + \frac{M}{L}$$

Check

Substituting in Eq. 01, we have

$$R_A + R_B = \frac{q \times L}{2} + \frac{M}{L} + \frac{q \times L}{2} - \frac{M}{L} = q\,L \text{ (O.K.)}$$

Zero Shear Force

Consider a section at a distance *x* where Shear Force is zero as shown in Fig. 3.55. From similar triangles, we have

$$\frac{\left[\frac{qL}{2} + \frac{M}{L}\right]}{x} = \frac{\left[\frac{qL}{2} - \frac{M}{L}\right]}{(L-x)} \text{ or } x = \left[\frac{L}{2} + \frac{M}{qL}\right]$$

$$= \frac{qL^2}{8} + \frac{M}{2} + \frac{M^2}{2qL^2}$$

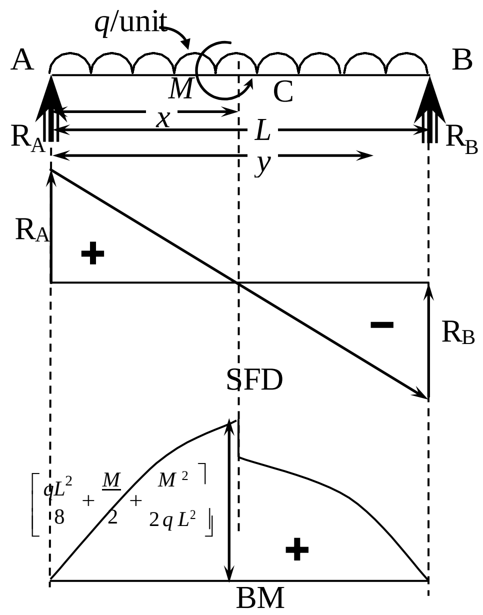


Fig. 3.55 SS with UDL &

Bending Moment Values

$M_B = 0$

$M_A = 0$

Bending Moment at zero Shear Force will be either Maximum or Minimum.

$$M_x = \left[\frac{q \times L}{2} + \frac{M}{L}\right] \times x - \frac{q}{2} \times x^2 = \left[\frac{q \times L}{2} + \frac{M}{L}\right] \times \left[\frac{L}{2} + \frac{M}{qL}\right] - \frac{q}{2} \times \left[\frac{L}{2} + \frac{M}{qL}\right]^2$$

$$M_{\max} = \frac{qL^2}{8} + \frac{M}{2} + \frac{M^2}{2qL^2}$$

3.17. For the loaded beam shown in Fig. 3.56, Draw the Shear Force and Bending Moment Diagram. Find and locate the Maximum +ve and —ve Bending Moments. Also locate the Point of Contraflexures. Detail the procedure to draw the SFD and BMD. (July 09)

It can be seen the loading is symmetrical and the Reactions are equal. From the conditions of equilibrium

$\sum V_A = 0;$

$$R_A + R_B = 2R_A = 2R_B = 2\times\left(20 + \frac{1}{2}\times(10\times 2) + 20\times 2\right) \text{ or } R_A = R_B = 50\text{kN}$$

Bending Moment Values

$M_F = M_C = 0$

$M_A = M_B = -20\times 2 = -40\text{kNm}$

$$M_{D_L} = M_{E_R} = 50\times 2 - 20\times 4 - \left(\frac{1}{2}\times 10\times 2\right)\left(\frac{2\times 2}{3}\right) = 6.67\text{kNm}$$

$$M_{D_R} = M_{E_L} = 6.67 - 10 = -3.33\text{kNm}$$

Maximum Bending Moment and Points of Contraflexure

Maxumum Bending Moment

Bending Moment at any section x in the region DE is given by

$$M_x = 50x - 20(x+2) - \left[\left(\frac{1}{2}\times 10\times 2\right)\left(x - \frac{2}{3}\right)\right] - 20\frac{(x-2)^2}{2} - 10$$

The Maximum bending moment occurs at zero shear force.

i.e. $x = (5\text{-}2) = 3$ m

$$M_x = 50\times 3 - 20(3+2) - \left[\left(\frac{1}{2}\times 10\times 2\right)\left(3 - \frac{2}{3}\right)\right] - 20\frac{(3-2)^2}{2} - 10 = 6.67\text{kNm}$$

Shear Force Diagram

1. Draw a horizontal line C_1F_2 equal to the length of the beam 10m to some scale, under the beam CF as shown.
2. Start the Shear force line from left extreme edge C_1. Draw C_1C_2 under the vertical load 20kN acting at C downward equal to some scale. To start with, the shear force at C_1=0 and at C_2, the Shear force = 0 – 20 (-ve as it is acting downward) = -20 kN.
3. There is no load in the region CA and hence under this region, the shear force line C_2A_1 will be a horizontal line parallel to beam axis.

4. At A, there is a reaction R_A which is treated as vertical load = 50kN and hence the shear force line A_1A_2 = 50kN to some scale and the shear force at A_2 = -20 + 50 (+ as it is upward) = +30kN.
5. There is a uvl in the region AD and the shear force line will be a parabola in this region. The parabola will be tangential to vertical at A_2 as there is relatively higher load intensity at A and will be parallel to horizontal at D_1 as the load intensity is lesser at D. Hence the curve is sagging. The vertical distance from A_2 to D_1 is equal to the total load equivalent to uvl, i.e. ½ x 10 x 2 = 10kN and the shear force at D_1 = 30 - 10 (- as it is downward) = +20 kN.
6. There is an udl in the region DE and hence the shear force line is inclined from D_1 to E_1. The vertical distance from D_1 to E_1 is equal to the total load equivalent to udl, i.e. 20 x 2 = 40kN and the shear force at E_1 = 20 - 40 (- as it is downward) = -20 kN.
7. There is a uvl in the region EB and the shear force line will be a parabola in this region. The parabola will be tangential to horizontal at E_1 as there is relatively lower load intensity at E and will be parallel to vertical at B_1 as the load intensity is higher at B. Hence the curve is hogging. The vertical distance from E_1 to B_1 is

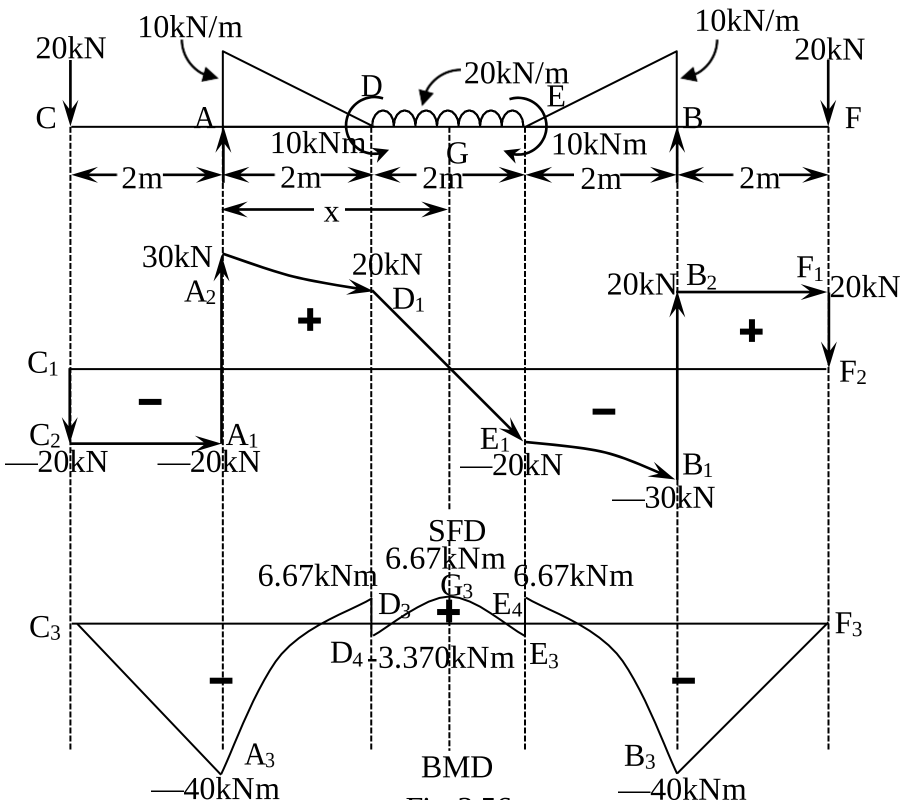


Fig. 3.56

equal to the total load equivalent to uvl, i.e. ½ x 10 x 2 = 10kN and the shear force at B_1 = -20 - 10 (- as it is downward) = -30 kN.

8. At B, there is a reaction R_B which is treated as vertical load = 50kN and hence the shear force line B_1B_2 = 50kN to same scale and the shear force at B_2 = -30 + 50 (+ as it is upward) = +20kN.
9. There is no load in the region BF and hence under this region, the shear force line B_2F_1 will be a horizontal line parallel to beam axis.
10. Draw F_1F_2 under the vertical load 20kN acting at F downward equal to same scale. The shear force at F_2 = 20 – 20 = 0 (-ve as it is acting downward). Note that for the Shear Force Diagram to be precise, the shear force line must finally join the horizontal axis. If there is any shortage or surplus, the shear force diagram must be redrawn.
11. The portion of the shear force diagram above the horizontal axis is +ve and the one below the horizontal axis is –ve.

Bending Moment Diagram

1. The Bending Moment is zero at the extreme edges of the beam unless there is an applied moment or couple acting at the edges, Hence the Moment at C = M_C = 0 i.e. at C_3.
2. The Bending moment at A is -40 kNm and hence the bending moment line is inclined under the no load portion CA (it can be either horizontal or inclined depending on the moments at the corresponding ends of the portion in the region).
3. The region AD has a uvl and hence the bending moment line will be a cubic parabola (the index of BM is always one more than SF at any section and hence bending moment line is inclined under horizontal shear force line, parabola under inclined shear force line and cubic parabola under parabolic shear force line). The parabola joins the bending moment values at A_3 is -40kNm and at D_3 is +6.67kNm (Bending moment to the left of D). The cubic parabola will be parallel to vertical at A_3 and parallel to horizontal at D_3 as the absolute value of shear force at A_2 = 30kN (more) compared to that at D_1 = 20kN.
4. The bending moment line is always a vertical line under the applied moment or couple. There is an clockwise applied moment of 10kNm acting at D and hence it is hogging. The vertical line D_3D_4 is downward and equal to the applied

moment to the same scale = 10kNm. The Bending moment value at D_4 = -3.37 kNm

5. The region DG is acted upon by udl, the shear force line is inclined and the bending moment line will be a parabola from D_4 to G_3. The parabola is joining Bending moment at D_4 = -3.37 to that at G_3 = 6.67kNm. The bending moment line will be tangential to vertical at D_4 and tangential to horizontal at G_3 as the shear force at D_1 = 20kN which is relatively higher than at G which is 0.
6. The region GE is acted upon by udl, the shear force line is inclined and the bending moment line will be a parabola from G_3 to E_3. The parabola is joining Bending moment at G_3 = 6.67 to that at E_3 = -3.37kNm. The bending moment line will be tangential to horizontal at G_3 and tangential to vertical at E_3 as the absolute shear force at G = 0kN which is relatively lesser than at E_3 =3.37kNm.
7. There is an anti-clockwise applied moment of 10kNm acting at E and hence it is sagging. The vertical line E_3E_4 is upward and equal to the applied moment to the same scale = 10kNm. The Bending moment value at E_4 = 6.67 kNm
8. The region EB has a uvl and hence the bending moment line will be a cubic parabola. The parabola joins the bending moment values at E_4 is 6.67kNm (Bending moment to the right of E) and at B_3 is -40kNm. The cubic parabola will be tangential to horizontal at E_4 and parallel to vertical at B_3 as the absolute value of shear force at E_1 = 20kN (less) compared to that at B_1 = 30kN.
9. The Bending moment at B is -40 kNm and hence the bending moment line is inclined under the no load portion BF to join the horizontal axis at F_3 where the bending moment is zero.

3.19 Draw the shear force and bending moment diagrams for a overhanging beam shown in Fig. 3.57. Find and locate the points of contraflexure. (July 09)

The reactions can be obtained from the conditions of equilibrium.

$$\sum V_A = 0;\ R_B + R_D = 10\times 2 + 40 + \frac{1}{2}\times 20\times 2 + 20 = 100\text{kN} \qquad (01)$$

Taking moment about B,

$$\Sigma M_B = 0;\ 4R_D + (10\times 2)\left(\frac{2}{2}\right) = 40\times 2 + \left(\frac{1}{2}\times 20\times 2\right)\left(2 + \frac{2\times 2}{3}\right) + 20\times 6$$

$$R_D = \frac{246.67}{4} = 61.67\text{kN}$$

Similarly taking moment about D,

$$\Sigma M_D = 0; 4R_B + (20\times 2) = (10\times 2)\left(4 + \frac{2}{2}\right) + 40\times 2 + \left(\frac{1}{2}\times 20\times 2\right)\left(\frac{2\times 1}{3}\right)$$

$$R_B = \frac{153.33}{4} = 38.33\text{kN}$$

Check

Substituting in Eq. 01, we have $R_B + R_D$ = 38.33 + 61.67 =100 kN (O.K.)

Bending Moment Values

$M_E = 0$

$$M_D = -20\times 2 = -40\text{kN}$$

$$M_C = 61.67\times 2 - 20\times 4 - \left(\frac{1}{2}\times 20\times 2\right)\left(\frac{2\times 2}{3}\right) = 16.67\text{kNm}$$

$$M_B = -(10\times 2)\left(\frac{2}{2}\right) = -20\text{kNm}$$

$M_A = 0$

Points of Contraflexures

Bending moment at any section x from the left support

For region CD

$$M_x = 38.33x - (10\times 2)(x+1) - 40(x-2) - \left(\frac{1}{2}\times 20\times\frac{(x-2)^2}{2}\right)\left(\frac{1}{3}(x-2)\right)$$

For Point of contraflexure, $M_x = 0$, solving, we get x = 2.713m

For region BC $M_x = 38.33x - (10\times 2)(x+1)$

For Point of contraflexure, $M_x = 0$, solving, we get x = 1.09m

From second method, consider the similar triangles between BC,

$$\frac{x}{20} = \frac{2-x}{16.67} \text{ or } x = 1.09\text{m}$$

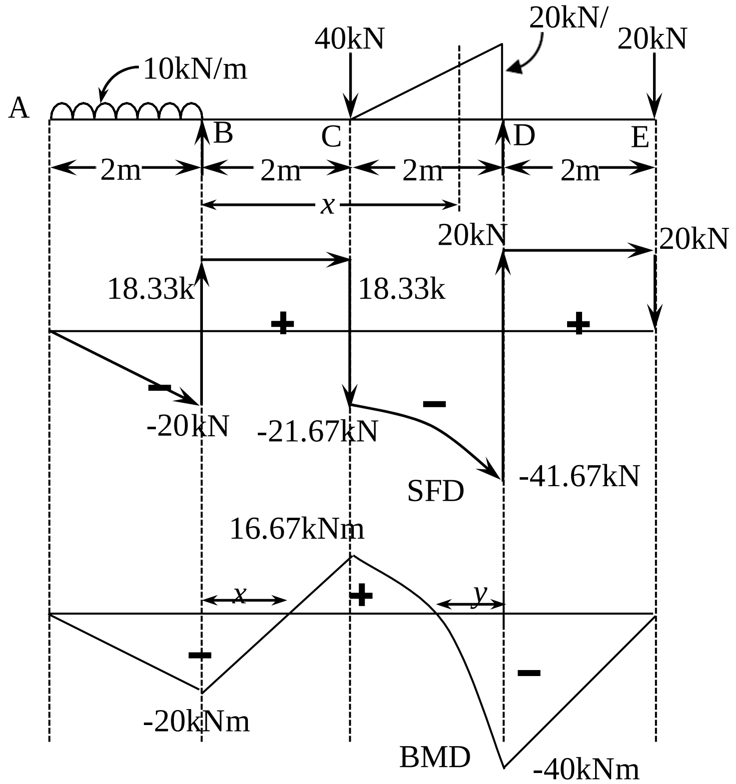


Fig. 3.57

3.20 For the beam shown in Fig.3.58, draw the shear force and bending moment diagram and locate the Point of contraflexure if any. (Jan 09)

The reactions can be obtained from the conditions of equilibrium.

$\sum V_A = 0;\ R_B + R_D = 10\times2+30+40+20\times4 = 170\text{kN}$ (01)

Taking moment about B,

$$\Sigma M_B = 0;\ 6R_D = (10\times2)\left(\frac{2}{2}\right) + 30\times2 + 40\times4 + (20\times4)\left(4+\frac{4}{2}\right) \text{ or } R_D = \frac{720}{6} = 120\text{kN}$$

Similarly taking moment about D,

$$\Sigma M_D = 0;\ 6R_B = (10\times2)\left(4+\frac{2}{2}\right) + 30\times4 + 40\times2 \text{ or } R_B = \frac{300}{6} = 50\text{kN}$$

Check

Substituting in Eq. 01, we have $R_B + R_D = 50 + 120 = 170$ kN (O.K.)

Bending Moment Values

$M_E = 0$

$$M_D = -(20\times2)\left(\frac{2}{2}\right) = -40\text{kN}$$

$$M_C = 120 \times 2 - (20 \times 4)\left(\frac{4}{2}\right) = 80\text{kNm}$$

$$M_B = 50 \times 2 - (10 \times 2)\left(\frac{2}{2}\right) = 80\text{kNm}$$

$M_A = 0$

Points of Contraflexures

Bending moment at any section x from the left support

For region CD

$$M_x = 38.33x - (10\times 2)(x+1) - 40(x-2) - \left(\frac{1}{2}\times 20\times \frac{(x-2)^2}{2}\right)\left(\frac{2}{3}\right)(x-2)$$

For Point of contraflexure, $M_x = 0$, solving, we get $x = 2.713$m

For region BC $M_x = 38.33x - (10 \times 2)(x+1)$

For Point of contraflexure, $M_x = 0$, solving, we get $x = 1.09$m

From second method, consider the similar triangles between BC,

$\frac{x}{20} = \frac{2 - x}{16.67}$ or $x = 1.09$m

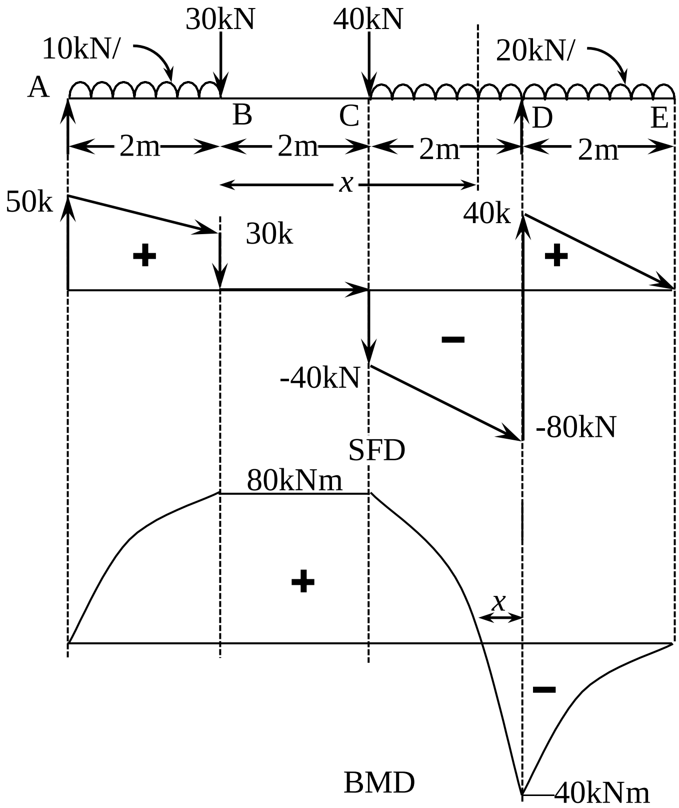


Fig. 3.58

3.21 For the beam shown in Fig. 3.59, obtain SFD and BMD. Locate Points of contraflexure, if any. (July 09)

The reactions can be obtained from the conditions of equilibrium.

$\sum V_A = 0;\ R_B + R_D = 5\times 8 + 50 = 90\text{kN}$ (01)

Taking moment about B,

$$\Sigma M_B = 0; 16R_D + 120 = (5\times 8)\left(\frac{8}{2}\right) + 50\times 12 + 160 \text{ or } R_D = \frac{800}{16} = 50\text{kN}$$

Similarly taking moment about D,

$$\Sigma M_D = 0; 16R_B + 160 = (5\times 8)\left(8 + \frac{8}{2}\right) + 50\times 4 + 120 \text{ or } R_D = \frac{640}{16} = 40\text{kN}$$

Check

Substituting in Eq. 01, we have $R_B + R_D = 40 + 50 = 90$ kN (O.K.)

Bending Moment Values

$M_{DR} = 0$

$M_{AL} = -160\text{kNm}$

$M_C = 50\times 4 - 160 = 40\text{kNm}$

$M_B = 50\times 8 - 50\times 4 - 160 = 40\text{kNm}$

$M_{AR} = -120\text{kNm}$

$M_{AL} = 0$

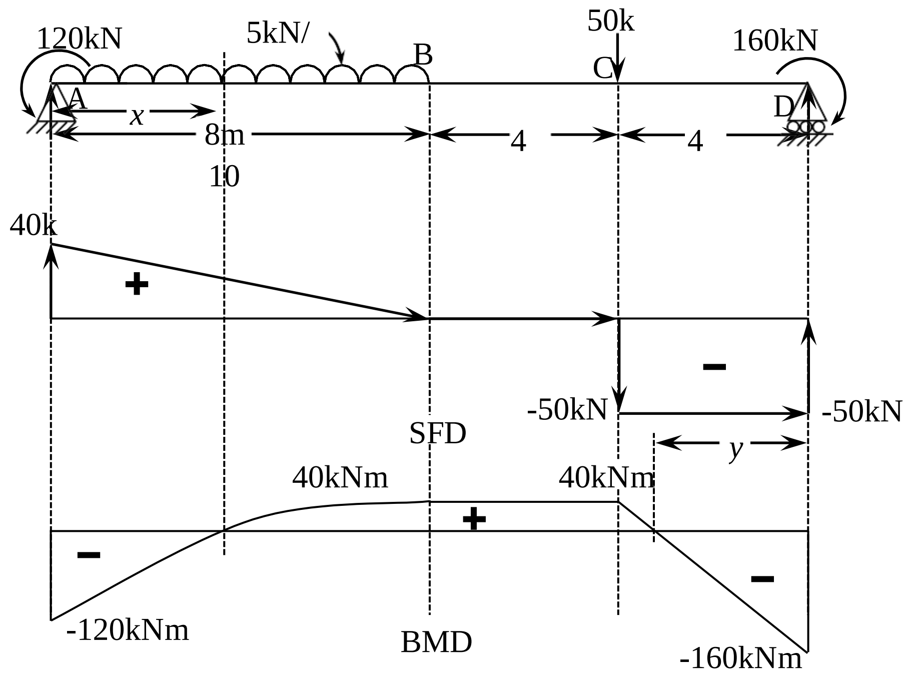


Fig. 3.59

Points of Contraflexures

Bending moment at any section x from the left support

For region AB

$$M_x = 40x - \left(\frac{5x^2}{2}\right) - 120 = 0 \text{ or } x = 4\text{m}$$

Point of contraflexure is $x = 4$m from the left support.

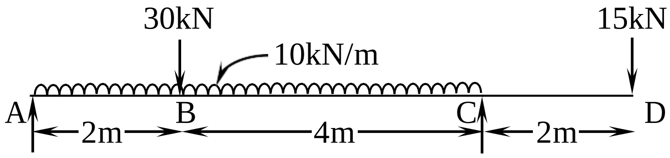
For region CD $M_y = 50y - 160 = 0$ or $y = 3.2$m

For Point of contraflexure is $y = 3.2$m from the right support.

From second method, consider the similar triangles between CD

$\frac{y}{160} = \frac{4-y}{40}$ or $y = 3.2$m

A beam ABCD, 8m long has supports at **A** and at **C** which is 6m from point **A.** The beam carries a **UDL** of 10kN/m between **A** and **C.** At point **B** a 30kN concentrated load acts 2m from the support **A** and a point load of 15kN acts at the free end **D**. Draw the SFD and BMD giving salient values. Also locate the point of contra-flexure if any. (14)(July 2015)



From the conditions of equilibrium, we have algebraic sum of vertical forces to be zero.

$\uparrow +\Sigma V = 0;\quad R_A + R_C = 30 + 15 + (10)(6) = 105 \text{ kN } (\uparrow)$

Algebraic sum of moments about any point is zero. Taking moments about A, we get

$$\Sigma M_A = 0;\quad 6R_C = (30)(2) + (15)(8) + \left[(10)(6)\right]\left(\frac{6}{2}\right) = 360 \text{ kN}$$

$$R_C = 60\text{kN}(\uparrow)$$

Taking moments about C, we get

$$\Sigma M_C = 0;\quad 6R_A + (15)(2) = (30)(4) + \left[(10)(6)\right]\left(\frac{6}{2}\right) = 270 \text{ kN}$$

$R_A = 45 \text{ kN}(\uparrow)$

Check: $R_A + R_C = 45 + 60 = 105 \text{ kN } (\uparrow)$

Shear Force Diagram can be directly drawn.

Bending Moment values:

Unless there are end moments of the beam, the Moments are zero at ends of the beam.

$M_A = 0$ and $M_D = 0$

$$M_B = (45)(2) - \left[(10)(2)\right]\left(\frac{2}{2}\right) = 70\text{kNm}$$

$$M_C = -(15)(2) = -30\text{kNm}$$

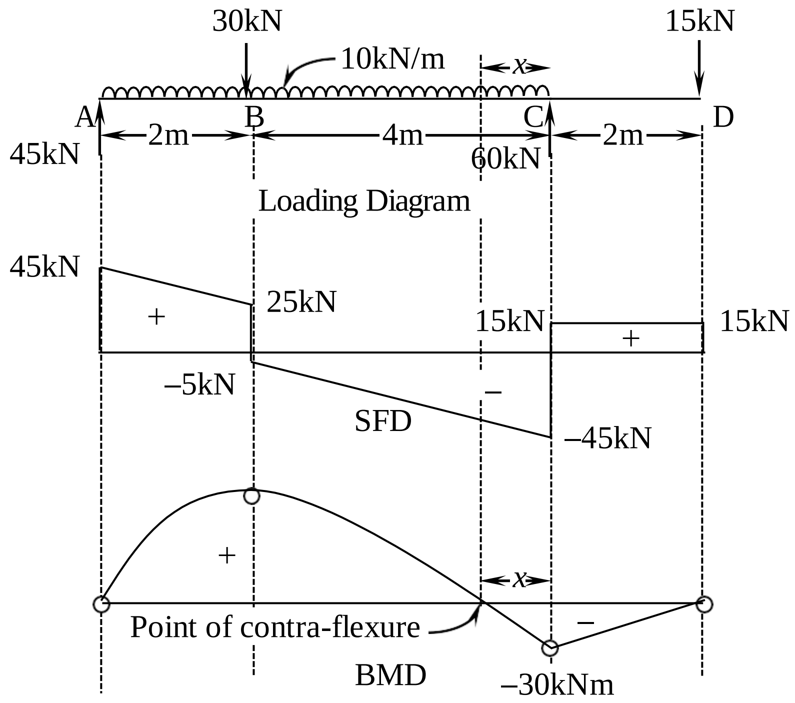
To locate the point of contra-flexure where the bending moment changes its sign, consider the section to be at a distance x towards left of the right support as shown. The bending moment at the section is given by

$$M_x = 60x - (15)(2+x) - (10)(x)\left(\frac{x}{2}\right) \Rightarrow 0$$

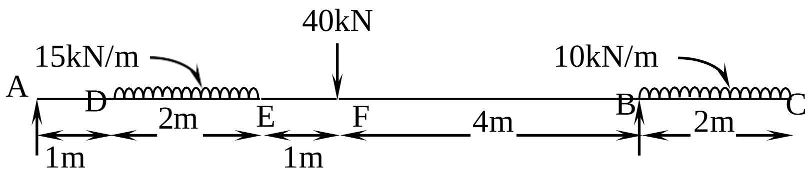
$45x - 30 - 5x^2 = 0$

Solving, $x = 0.725$m and 8.275m

Hence the point of contra-flexure is at 0.725m to left of right support.



Draw the Shear force and bending moment diagrams for the Fig. shown (10) July 2016



From the conditions of equilibrium, we have algebraic sum of vertical forces to be zero.

$\uparrow +\Sigma V = 0;$ $\quad R_A + R_B = (15)(2) + 40 + (10)(2) = 90 \text{ kN } (\uparrow)$

Algebraic sum of moments about any point is zero. Taking moments about A, we get

$\Sigma M_A = 0;$ $\quad 8R_B = [(15)(2)]\left(1+\frac{2}{2}\right) + (40)(1+2+1) + [(10)(2)]\left(8+\frac{2}{2}\right) = 400 \text{ kN}$

$R_B = 50\text{kN}(\uparrow)$

Taking moments about B, we get

$\Sigma M_B = 0;\qquad 8R_A + \left[(10)(2)\right]\left(\frac{2}{2}\right) = (40)(4) + \left[(15)(2)\right]\left(4+1+\frac{2}{2}\right) = 340\ \text{kN}$

$R_A = 40\text{kN}(\uparrow)$

Check: $R_A + R_B = 40 + 50 = 90$ kN $(\uparrow)$

Shear Force Diagram can be directly drawn.

Bending Moment values:

Unless there are end moments of the beam, the Moments are zero at ends of the beam.

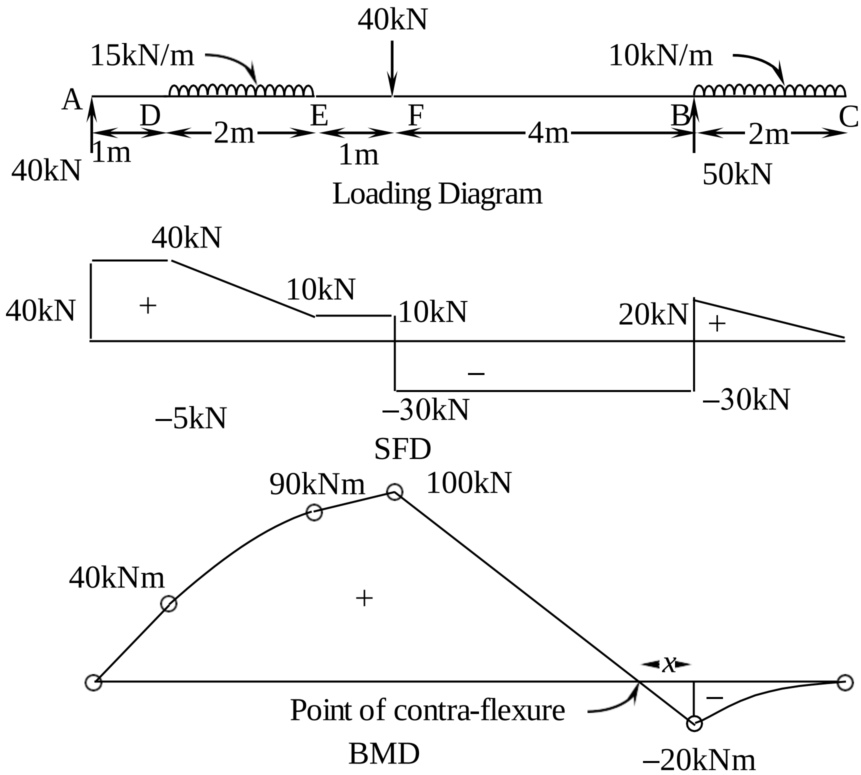
$M_A = 0$ and $M_C = 0$

$M_D = (40)(1) = 40\text{kNm}$

$M_E = (40)(3) - \left[(15)(2)\right]\left(\frac{2}{2}\right) = 90\text{kNm}$

$M_F = (40)(4) - \left[(15)(2)\right]\left(1+\frac{2}{2}\right) = 100\text{kNm}$

$M_B = -\left[(10)(2)\right]\left(\frac{2}{2}\right) = -20\text{kNm}$

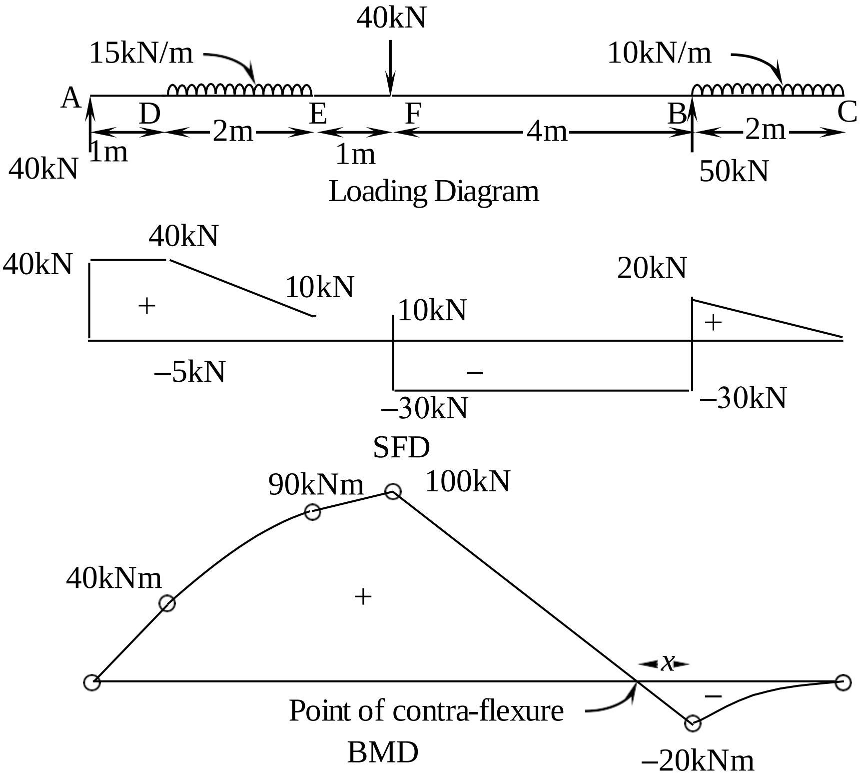


To locate the point of contra-flexure where the bending moment changes its sign, consider the section to be at a distance x towards left of the right support as shown. Bending moment inclined line is crossing zero line as a straight line forming two alternate triangles which are similar. Hence using similar triangle properties

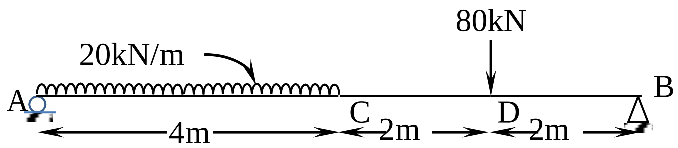
$$\frac{4-x}{x}=\frac{100}{20}$$

Solving, $x = 0.67\text{m}$

Hence the point of contra-flexure is at 0.67m to left of right support.



Draw Shear force and Bending moment Diagram for the beam shown in Fig.



Fromthe conditions of equilibrium, we have algebraic sum of vertical forces to be zero.

$\uparrow + \Sigma V = 0;$ $\quad R_A + R_B = (20)(4) + 80 = 160 \text{ kN } (\uparrow)$

$\Sigma M_A = 0;$ $\quad 8R_B = \left[(20)(4)\right]\left(\frac{4}{2}\right) + (80)(4+2) = 640 \text{ kN}$

$R_B = 80 \text{ kN}(\uparrow)$

Algebraic sum of moments about any point is zero. Taking moments about A, we get

$$\Sigma M_A = 0; \qquad 8R_B = [(20)(4)]\left(\frac{4}{2}\right) + (80)(4+2) = 640 \text{ kN}$$

$$R_B = 80 \text{ kN}(\uparrow)$$

Taking moments about B, we get

$$\Sigma M_B = 0; \qquad 8R_A = [(20)(4)]\left(4+\frac{4}{2}\right) + (80)(2) = 640 \text{ kN}$$

$$R_A = 80 \text{ kN}(\uparrow)$$

Check: $R_A + R_B = 80 + 80 = 160 \text{ kN } (\uparrow)$

Shear Force Diagram can be directly drawn.

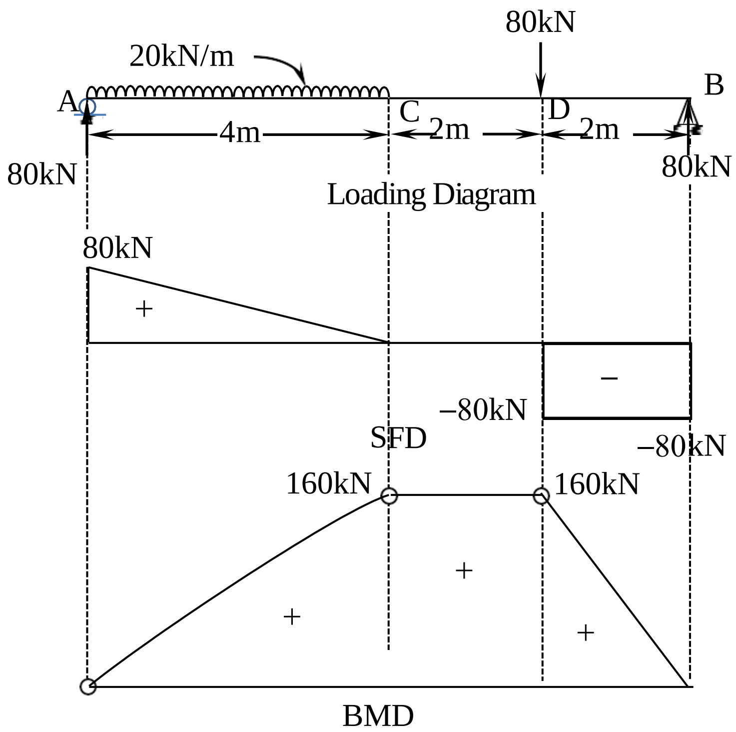
Bending Moment values:

Unless there are end moments of the beam, the Moments are zero at ends of the beam.

$M_A = 0$ and $M_B = 0$

$$M_C = (80)(4) - [(20)(4)]\left(\frac{4}{2}\right) = 160\text{kNm}$$

$M_D = (80)(2) = 160\text{kNm}$



Module 5: Theories of Failure

Objectives:

The objectives/outcomes of this lecture on “Theories of Failure” is to enable students for

1. Recognize loading on Structural Members/Machine elements and allowable stresses.
2. Comprehend the Concept of yielding and fracture.
3. Comprehend Different theories of failure.
4. Draw yield surfaces for failure theories.
5. Apply concept of failure theories for simple designs

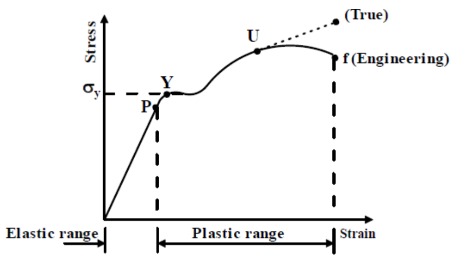
1. Introduction:

Failure indicate either fracture or permanent deformation beyond the operational range due to yielding of a member. In the process of designing a machine element or a structural member, precautions has to be taken to avoid failure under service conditions.

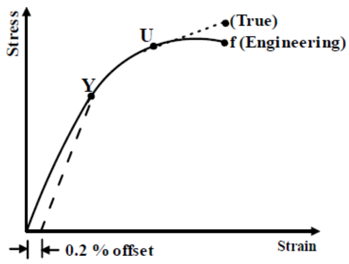
When a member of a structure or a machine element is subjected to a system of complex stress system, prediction of mode of failure is necessary to involve in appropriate design methodology. Theories of failure or also known as failure criteria are developed to aid design.

1.1 Stress-Strain relationships:

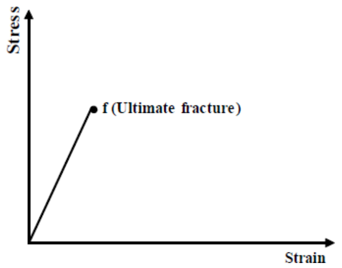
Following Figure-1 represents stress-strain relationship for different type of materials.



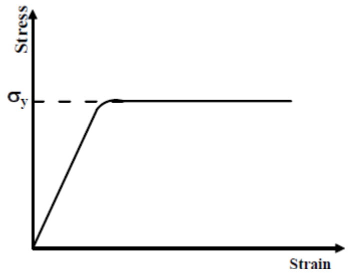
Ductile material e.g. low carbon steel



Low ductility



Brittle material



Elastic – perfectly plastic material

Figure-1: Stress-Strain Relationship

Bars of ductile materials subjected to tension show a linear range within which the materials exhibit elastic behaviour whereas for brittle materials yield zone cannot be identified. In general, various materials under similar test conditions reveal different behaviour. The cause of failure of a ductile material need not be same as that of the brittle material.

1.2 Types of Failure:

The two types of failure are,

Yielding - This is due to excessive inelastic deformation rendering the structural member or machine part unsuitable to perform its function. This mostly occurs in ductile materials.

Fracture - In this case, the member or component tears apart in two or more parts. This mostly occurs in brittle materials.

1.3 Transformation of plane stress:

For an element subjected to biaxial state of stress the normal stress on an inclined plane is determined as,

—— —— - Eq-1

Similarly, on the same inclined plane the value of the shear stress is determined as,

—— - Eq-2

The above equations (Eq-1 and Eq-2) are used to determine the condition when the normal stress and shear stress values are maximum/minimum by differentiating them with respect to θ and equating to zero. The substitution of the results in these equations determines maximum and minimum normal stress known as principal stresses and maximum shear stress as indicated by the following expressions (Eq-3 and Eq-4).

—— √(—)—— - Eq.-3

√(——) - Eq-4

1.4 <u>Use of factor of safety in design:</u>

In designing a member to carry a given load without failure, usually a factor of safety (FS or N) is used. The purpose is to design the member in such a way that it can carry N times the actual working load without failure. Factor of safety is defined as Factor of Safety (FS) = Ultimate Stress/Allowable Stress.

2. Theories of Failure:

a) Maximum Principal Stress Theory (Rankine Theory)
b) Maximum Principal Strain Theory (St. Venant's theory)
c) Maximum Shear Stress Theory (Tresca theory)
d) Maximum Strain Energy Theory (Beltrami's theory)

2.1 Maximum Principal Stress Theory (Rankine theory)

According to this, if one of the principal stresses σ_1 (maximum principal stress), σ_2 (minimum principal stress) or σ_3 exceeds the yield stress (σ_y), yielding would occur. In a two dimensional loading situation for a ductile material where tensile and compressive yield stress are nearly of same magnitude

$\sigma_1 = \pm \sigma_y$ $\sigma_2 = \pm\sigma_y$

Yield surface for the situation is, as shown in Figure-2

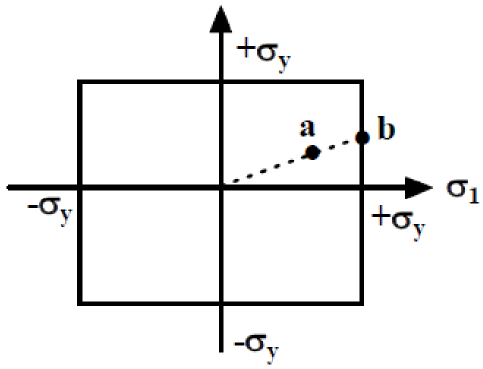


Figure- 2: Yield surface corresponding to maximum principal stress theory

Yielding occurs when the state of stress is at the boundary of the rectangle. Consider, for example, the state of stress of a thin walled pressure vessel. Here $\sigma_1 = 2\sigma_2$, σ_1 being the circumferential or hoop stress and σ_2 the axial stress. As the pressure in the vessel increases, the stress follows the dotted line. At a point (say) a, the stresses are still within the elastic limit but at b, σ_1 reaches σ_y although σ_2 is still less than σ_y. Yielding will then begin at point b. This theory of yielding has very poor agreement with experiment. However, this theory is being used successfully for brittle materials.

2.2 Maximum Principal Strain Theory (St. Venant's Theory)

According to this theory, yielding will occur when the maximum principal strain just exceeds the strain at the tensile yield point in either simple tension or compression. If ε_1 and ε_2 are maximum and minimum principal strains corresponding to σ_1 and σ_2, in the limiting case

$\varepsilon_1 = (1/E)(\sigma_1 - \nu\sigma_2)$ $|\sigma_1| \geq |\sigma_2|$

$\varepsilon_2 = (1/E)(\sigma_2 - \nu\sigma_1)$ $|\sigma_2| \geq |\sigma_1|$

This results in,

$E\,\varepsilon_1 = \sigma_1 - \nu\sigma_2 = \pm\,\sigma_0$
$E\,\varepsilon_2 = \sigma_2 - \nu\sigma_1 = \pm\,\sigma_0$

The boundary of a yield surface in this case is shown in Figure – 3.

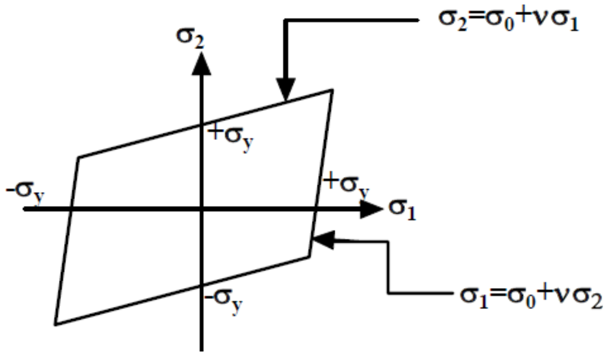


Figure-3: Yield surface corresponding to maximum principal strain theory

2.3 Maximum Shear Stress Theory (Tresca theory)

According to this theory, yielding would occur when the maximum shear stress just exceeds the shear stress at the tensile yield point. At the tensile yield point $\sigma_2 = \sigma_3 = 0$ and thus maximum shear stress is $\sigma_y/2$. This gives us six conditions for a three-dimensional stress situation:

$\sigma_1 - \sigma_2 = \pm\,\sigma_y$
$\sigma_2 - \sigma_3 = \pm\,\sigma_y$
$\sigma_3 - \sigma_1 = \pm\,\sigma_y$

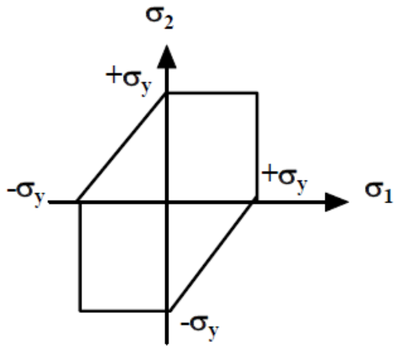


Figure – 4: Yield surface corresponding to maximum shear stress theory

In a biaxial stress situation (Figure - 4) case, $\sigma_3 = 0$ and this gives

$\sigma_1 - \sigma_2 = \sigma_v$	if $\sigma_1 > 0, \sigma_2 < 0$
$\sigma_1 - \sigma_2 = -\sigma_v$	if $\sigma_1 < 0, \sigma_2 > 0$
$\sigma_2 = \sigma_v$	if $\sigma_2 > \sigma_1 > 0$
$\sigma_1 = -\sigma_v$	if $\sigma_1 < \sigma_2 < 0$
$\sigma_1 = -\sigma_v$	if $\sigma_1 > \sigma_2 > 0$
$\sigma_2 = -\sigma_v$	if $\sigma_2 < \sigma_1 < 0$

This criterion agrees well with experiment.

In the case of pure shear, $\sigma_1 = - \sigma_2 = k$ (say), $\sigma_3 = 0$
and this gives $\sigma_1 - \sigma_2 = 2k = \sigma_y$
This indicates that yield stress in pure shear is half the tensile yield stress and this is also seen in the Mohr's circle (Figure - 5) for pure shear.

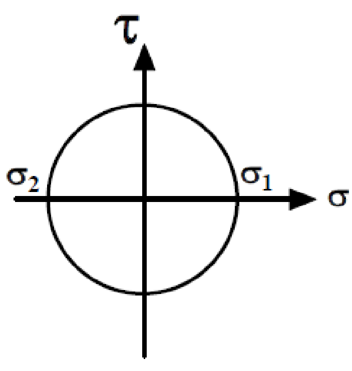


Figure – 5: Mohr's circle for

pure shear

2.4 Maximum strain energy theory (Beltrami's theory)

According to this theory failure would occur when the total strain energy absorbed at a point per unit volume exceeds the strain energy absorbed per unit volume at the tensile yield point. This may be expressed as,

$$(1/2)(\sigma_1\,\varepsilon_1 + \sigma_2\,\varepsilon_2 + \sigma_3\,\varepsilon_3) = (1/2)\,\sigma_y\,\varepsilon_y$$

Substituting ε_1, ε_2, ε_3 and ε_y in terms of the stresses we have

$$\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\,\upsilon\,(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1) = \sigma_y^2$$

$$(\sigma_1/\,\sigma_y)^2 + (\sigma_2/\,\sigma_y)^2 - 2\nu(\sigma_1\,\sigma_2/\,\sigma_y^2) = 1$$

The above equation represents an ellipse and the yield surface is shown in F igure - 6

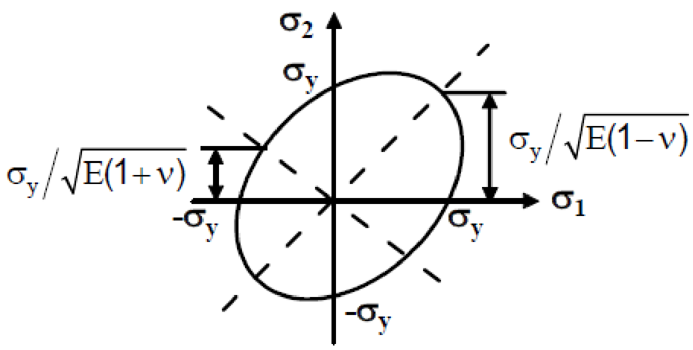


Figure – 6: Yield surface corresponding to Maximum strain energy theory.

It has been shown earlier that only distortion energy can cause yielding but in the above expression at sufficiently high hydrostatic pressure $\sigma_1 = \sigma_2 = \sigma_3 = \sigma$ (say), yielding may also occur. From the above we may write $\sigma^2(3 - 2\nu) = \sigma_y^2$ and if $\nu \sim 0.3$, at stress level lower than yield stress, yielding would occur. This is in contrast to the experimental as well as analytical conclusion and the theory is not appropriate.

2.5 Superposition of yield surfaces of different failure theories:

A comparison among the different failure theories can be made by superposing the yield surfaces as shown in figure – 7. It is clear that an immediate assessment of failure probability can be made just by plotting any experimental in the combined yield surface. Failure of ductile materials is most accurately governed by the distortion energy theory where as the maximum principal strain theory is used for brittle materials.

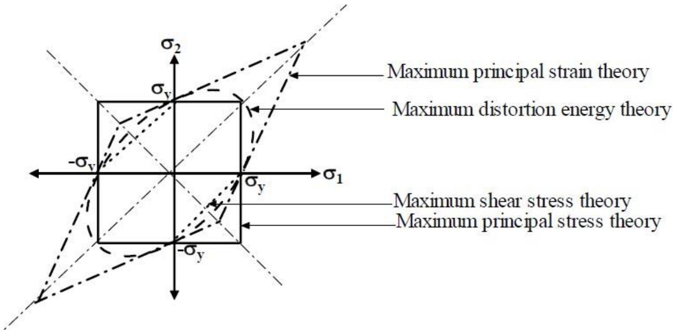


Figure – 7: Comparison of different failure theories

Numerical-1: A shaft is loaded by a torque of 5 KN-m. The material has a yield point of 350 MPa. Find the required diameter using Maximum shear stress theory. Take a factor of safety of 2.5.

Torsional Shear Stress, $\tau = 16T/\pi d^3$, where d represents diameter of the shaft

Maximum Shear Stress theory, √ ———— (——)

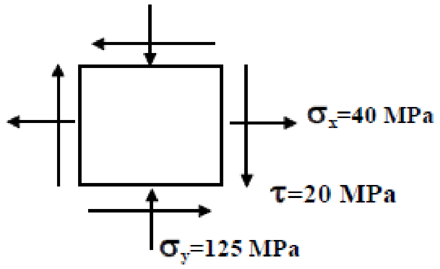
Factor of Safety (FS) = Ultimate Stress/Allowable Stress

Since $\sigma_x = \sigma_y = 0$, $\tau_{max} = 25.46 \text{ X } 10^3/d^3$

Therefore $25.46 \text{ X } 10^3/d^3 = \sigma_y/(2*FS) = 350*10^6/(2*2.5)$

Hence, d = 71.3 mm

Numerical-2: The state of stress at a point for a material is shown in the following figure Find the factor of safety using (a) Maximum shear stress theory Take the tensile yield strength of the material as 400 MPa.



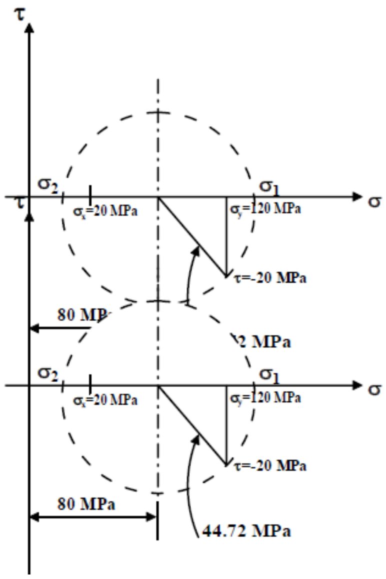
From the Mohr's circle shown below we determine,

σ_1 = 42.38MPa and
σ_2 = -127.38MPa

from Maximum Shear Stress theory

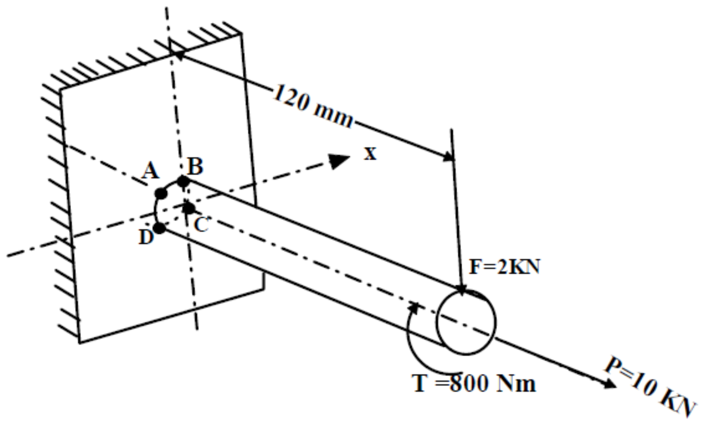
$(\sigma_1 - \sigma_2)/2 = \sigma_y/(2*FS)$

By substitution and calculation factor of safety FS = 2.356

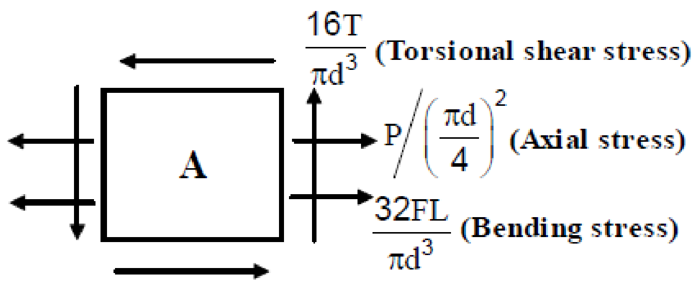


Numerical-3: A cantilever rod is loaded as shown in the following figure. If the tensile yield strength of the material is 300 MPa determine the

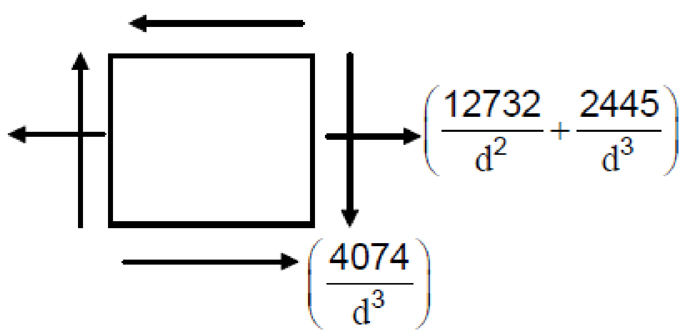
rod diameter using (a) Maximum principal stress theory (b) Maximum shear stress theory



At the outset it is necessary to identify the mostly stressed element. Torsional shear stress as well as axial normal stress is the same throughout the length of the rod but the bearing stress is largest at the welded end. Now among the four corner elements on the rod, the element A is mostly loaded as shown in following **figure**



Shear stress due to bending VQ/It is also developed but this is neglected due to its small value compared to the other stresses. Substituting values of T, P, F and L, the elemental stresses may be shown as in following **figure.**



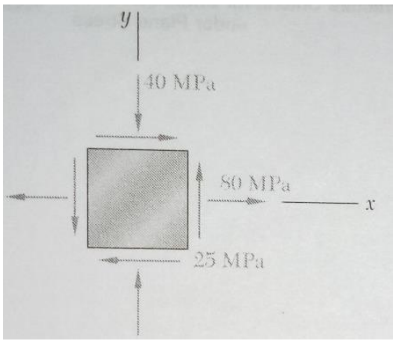
The principal stress for the case is determined by the following equation,

$$\sigma_{1,2} = \frac{1}{2}\left(\frac{12732}{d^2} + \frac{2445}{d^3}\right) \pm \sqrt{\frac{1}{4}\left(\frac{12732}{d^2} + \frac{2445}{d^3}\right)^2 + \left(\frac{4074}{d^3}\right)^2}$$

By Maximum Principal Stress Theory, Setting, $\sigma_1 = \sigma_y$ we get d = 26.67mm

By maximum shear stress theory by setting $(\sigma_1 - \sigma_2)/2 = \sigma_y/2$, we get, d = 30.63mm

Numerical-4: The state of plane stress shown occurs at a critical point of a steel machine component. As a result of several tensile tests it has been found that the tensile yield strength is σ_y=250MPa for the grade of steel used. Determine the factor of safety with respect to yield using maximum shearing stress criterion.

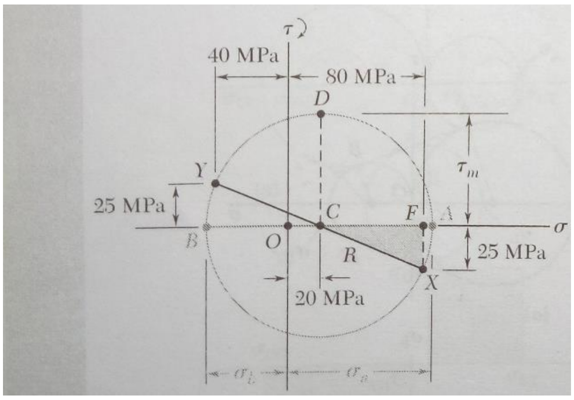


Construction of the Mohr's circle determines

σ_{avg} = ½ (80-40) = 20MPa and $\tau_m = (60^2+25^2)^{1/2}$ = 65MPa
σ_a= 20+65 = 85 MPa and σ_b= 20-65 = -45 MPa

The corresponding shearing stress at yield is τ_y= ½ σ_y = ½ (250) = 125MPa

Factor of safety, FS = τ_m / τ_y = 125/65 = 1.92

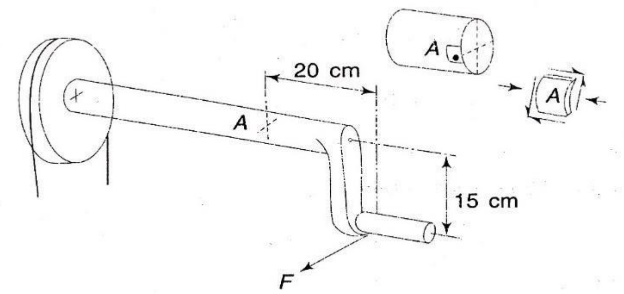


Summary:

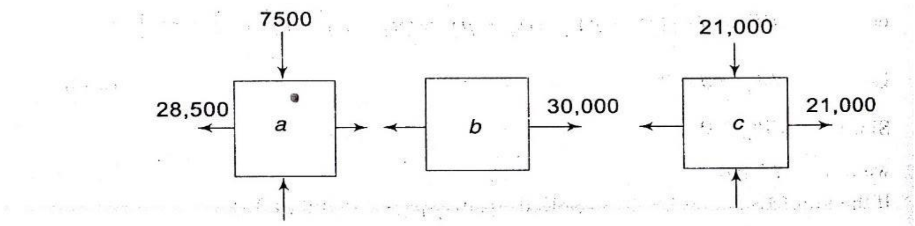
Different types of loading and criterion for design of structural members/machine parts subjected to static loading based on different failure theories have been discussed. Development of yield surface and optimization of design criterion for ductile and brittle materials were illustrated.

Assignments:

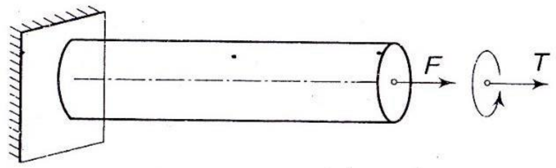
Assignment-1: A Force F = 45,000N is necessary to rotate the shaft shown in the following figure at uniform speed. The crank shaft is made of ductile steel whose elastic limit is 207,000 kPa, both in tension and compression. With E = 207 X 10^6 kPa and ν = 0.25, determine the diameter of the shaft using maximum shear stress theory, using factor of safety = 2. Consider a point on the periphery at section A for analysis **(Answer, d = 10.4 cm)**



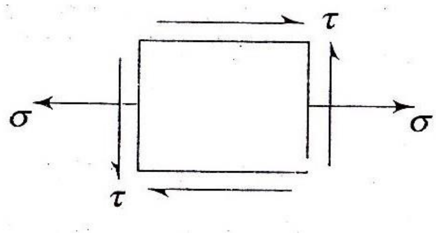
Assignment-2: Following figure shows three elements a, b and c subjected to different states of stress. Which one of these three, do you think will yield first according to i) maximum stress theory, ii) maximum strain theory, and iii) maximum shear stress theory? Assume Poisson's ratio ν = 0.25 **[Answer: i) b, ii) a, and iii) c]**



Assignment-3: Determine the diameter of a ductile steel bar if the tensile load F is 35,000N and the torsional moment T is 1800N.m. Use factor of safety = 1.5. E = $207*10^6$kPa and σ_{yp} = 207,000kPa. Use the maximum shear stress theory. **(Answer: d = 4.1cm)**



Assignment-4: At a pint in a steel member, the state of stress shown in Figure. The tensile elastic limit is 413.7kPa. If the shearing stress at a point is 206.85kPa, when yielding starts, what is the tensile stress σ at the point according to maximum shearing stress theory? **(Answer: Zero)**



Reference:

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